Level-Set Methods
Overview

• Level set methods
  – Formulation of Interface Propagation
    • Boundary Value PDE
    • Initial Value PDE
  – Motion in an externally generated velocity field
    • Convection
    • Upwind differencing
    • Hamilton-Jacobi ENO/WENO, TVD Runge-Kutta
  – Motion involves mean curvature
    • Equations of motion
    • Numerical discretization
    • Convection-diffusion equations
Overview

– Hamilton-Jacobi Equation
  • Connection with Conservation Laws
  • Numerical discretization

– Motion in the Normal Direction
  • The Basic Equation
  • Numerical discretization
  • Adding a Curvature-Dependent Term
  • Adding an External Velocity Field
References

• J.A. Sethian, Level Set Methods and Fast Marching Methods, Cambridge, 1996.
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3. Motion in an Externally Generated Velocity Field

1. Convection

• Assume the velocity of each point on the implicit surface is given as $\vec{V}(\vec{x})$,
• i.e. $\vec{V}(\vec{x})$ is known for every point $\vec{x}$ with $\phi(\vec{x}) = 0$
• Given this velocity field $\vec{V}(\vec{x})$
Motion in an Externally Generated Velocity Field

• Level set methods add dynamics to implicit surfaces

• Stanley Osher and James Sethian implemented the first approach to numerically solving a time-dependent equation for a moving implicit surface
Motion in an Externally Generated Velocity Field

• In the Eulerian approach the implicit function $\varphi$ is used to both represent the interface and evolve the interface.

• Given an external velocity field the level set equation is given as

$$\varphi_t + \vec{V} \cdot \nabla \varphi = 0$$

- Temporal partial derivative
- Advection term
Motion in an Externally Generated Velocity Field

• Using the forward difference approach for the partial derivative with respect to time

\[
\frac{\phi^{n+1} - \phi^n}{\Delta t} + \vec{V}^n \cdot \nabla \phi^n = 0
\]
Motion in an Externally Generated Velocity Field

• At this point it looks like the next step would be to take the gradient of $\phi$ by using central differences.

• But first we should expand the equation and look at it one dimension at a time

$$\frac{\phi^{n+1} - \phi^n}{\Delta t} + u^n \phi_x^n + v^n \phi_y^n + w^n \phi_z^n = 0$$
Motion in an Externally Generated Velocity Field

- If we look at the $u$ component of the vector field we might see something like to the right for direction.
- If $u_{ij}$ is pointing to the right we need to look to the cell to the right of $\varphi_{ij}$ the same applies to values to the left.

$$\frac{\partial \varphi}{\partial x} = \frac{\varphi_{i+1} - \varphi_i}{\Delta x} \quad \frac{\partial \varphi}{\partial x} = \frac{\varphi_i - \varphi_{i-1}}{\Delta x}$$

- Forward Difference
- Backward Difference
Motion in an Externally Generated Velocity Field

• For a 1-d case the level set equation would be of the form

\[
\frac{\phi^{n+1} - \phi^n}{\Delta t} + u_i^n \phi_x^n = 0
\]
Motion in an Externally Generated Velocity Field

• The update equation would be

\[ \phi^{n+1} = \phi^n + \Delta t u^n \phi_x^n \]

• where

\[ \phi_x = \begin{cases} \frac{\phi_{i+1} - \phi_i}{\Delta x} & u_i > 0 \\ \frac{\phi_i - \phi_{i-1}}{\Delta x} & u_i < 0 \end{cases} \]
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4. Motion involves mean curvature

- Equations of motion
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- Convection-diffusion equations
Figure 4.1. Evolution of a wound spiral in a curvature-driven flow. The high-curvature ends of the spiral move significantly faster than the elongated body section.
Figure 4.2. Evolution of a star-shaped interface in a curvature-driven flow. The tips of the star move inward, while the gaps in between the tips move outward.
Overview

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Hamilton-Jacobi Equation

• Connection with Conservation Laws
• Numerical discretization
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Motion in the Normal Direction

- The Basic Equation
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Figure 6.1. Evolution of a star-shaped interface as it moves normal to itself in the outward direction.
Figure 6.2. A star-shaped interface being advected by a rigid body rotation as it moves outward normal to itself.
Practical Animation of Liquids

• Foster and Fedkiw wanted to create a good animation tool that captured the essence of a good fluid simulator for animators

• Most engineering methods available were based around setting initial and boundary conditions and letting the fluid go with minimal ability for an animator to change anything dynamically
Practical Animation of Liquids

• Method Outline
  Setup
  1. Model static environment as a voxel grid
  2. Model the liquid volume as a combination of particles and an implicit surface

Repeat each iteration
  1. Update the velocity field
  2. Apply velocity constraints due to moving objects
  3. Enforce incompressibility
  4. Update the position of the liquid volume using the velocity field
Practical Animation of Liquids

- Model static environment as a voxel grid
  - The “staggered” grid shares velocity components between adjacent cells at the faces
  - Each cell is marked as either empty or full
Practical Animation of Liquids

- Model the liquid volume as a combination of particles and an implicit surface
  - Randomly generate particles with surfaces represented as implicit surfaces.
  - Generate an initial isocontour of the liquid by taking the minimum signed distance for each grid cell and smoothing it out.
Practical Animation of Liquids

• Hybrid Surface Model
  – The Level Set method alone tends to have problems preserving fine feature details like thin splashing
  – A layer of particles are maintained just inside the isocontour surface and are evolved forward in time using Lagrangian methods.
Practical Animation of Liquids

• Hybrid Surface Model
  – For each particle the local curvature of $\varphi$ is tested by

  \[ k = \nabla \cdot \left( \frac{\nabla \varphi}{|\nabla \varphi|} \right) \]

  – For areas of low curvature the particle is ignored but for
    areas of high curvature the particle is allowed to modify $\varphi$
  – Particles that escape to the exterior region can be
    visualized as water droplets or mist
Practical Animation of Liquids

- Update the velocity field
  - Performed using Stam’s formulation of Navier-Stokes
  - Enough said.
Practical Animation of Liquids

- Apply velocity constraints due to moving objects
  1. Any cell within a solid object has its velocities changed to that of the moving object
  2. Apply the velocity update
  3. Each cell that intersects the surface of the object gets its velocity set to be tangent to the surface of the object
  4. Cells inside the object have their velocities put back to the object velocity
Practical Animation of Liquids

• Enforce incompressibility
  – Last step of Navier-Stokes update

• Update the position of the liquid volume using the velocity field
  – Apply the level set equation to the liquid field

\[ \phi_t + \vec{u} \cdot \nabla \phi = 0 \]
Results
Implicit Functions

- A discrete approximation of the implicit function can be achieved via a Cartesian grid.

\[ \phi(x) = x^2 + y^2 - 1 = 0 \]

\[ \partial \Omega \]

Outside \( \Omega^+ \) \( \phi > 0 \)

Inside \( \Omega^- \) \( \phi < 0 \)
Implicit Functions

- The finer the resolution of the grid the more numerically accurate the model will be.

\[ \varphi(x) = x^2 + y^2 - 1 = 0 \]

\[ \partial \Omega \]

Outside \( \Omega^+ \)
\[ \varphi > 0 \]

Inside \( \Omega^- \)
\[ \varphi < 0 \]
Implicit Vs. Explicit

• Splitting and Merging

The merge of two implicit functions is the union operation.
Implicit Vs. Explicit

- Expansion and Contraction

Although there is no need to resample the region there are prescribed limits of expansion and contractions based on initial grid size.
Implicit Functions

• Important Realizations
  – The Cartesian grid represents the evolving implicit function \( \varphi(x) \).
  – It may begin as the reflection of the form of a circle or sphere or any shape but will soon represent something else entirely.
  – The only relation to its original form it must maintain is \( \varphi(x) = 0 \) represents the interface.
Implicit Functions

• Important Realizations
  – The interface evolves as elements of the grid become less positive or negative.
  – It is best not to think in terms of the interface itself moving but that the grid cells around it are getting values closer to 0.
Implicit Functions

• Example
  – Using an active contour to segment an image
Implicit Functions: Discrete Representations

- **Points**
  - Another way to represent $\phi(x)$ is to discretize $\Omega$ into grid cells and put the value of $\phi(x)$ in each cell.

$$\phi(x) = x^2 - 1$$

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</tr>
</tbody>
</table>
Implicit Functions

- Points
  - Or define $\varphi(x)$ as a height.

$$\varphi(x) = x^2 - 1$$
Implicit Surface & Isosurface

• When $c$ is 0, we say that $f$ implicitly defines a locus, called an implicit surface, $\{p \in \mathbb{R}^3 \mid f(p) = c\}$

• The implicit surface is sometimes called the zero set (or zero surface) of $f$.

• The isosurface (also called a level set or level surface) is $\{p \in \mathbb{R}^3 \mid f(p) = 0\}$ where $c$ is the iso-value of the surface.
Boolean Operations of Implicit Functions

- \( \varphi(x) = \min (\varphi_1(x), \varphi_2(x)) \) is the union of the interior regions of \( \varphi_1(x) \) and \( \varphi_2(x) \).

- \( \varphi(x) = \max (\varphi_1(x), \varphi_2(x)) \) is the intersection of the interior regions of \( \varphi_1(x) \) and \( \varphi_2(x) \).

- \( \varphi(x) = -\varphi_1(x) \) is the complement of \( \varphi_1(x) \).

- \( \varphi(x) = \max (\varphi_1(x), -\varphi_2(x)) \) represents the subtraction of the interior regions of \( \varphi_1(x) \) by the interior regions of \( \varphi_2(x) \).