

## CHAPTER 18

*Development of Mathematical Understanding*

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The nature of children's numerical, arithmetical, and mathematical understanding and the mechanisms that underlie the development of this knowledge are at the center of an array of scientific, political, and educational debates. Scientific issues range from infants' understanding of quantity and arithmetic (Cohen & Marks, 2002; Starkey, 1992; Wynn, 1992a) to the processes that enable middle-school and high-school students to solve multistep arithmetical and algebraic word problems (Tronsky & Royer, 2002). The proposed mechanisms that underlie quantitative and mathematical knowledge range from inherent systems that have been designed through evolution to represent and process quantitative information (Geary, 1995; Spelke, 2000; Wynn, 1995) to general learning mechanisms that can operate on and generate arithmetical and mathematical knowledge but are not inherently quantitative (Newcombe, 2002). The wide range of competencies covered under the umbrella of children's mathematical

understanding and the intense scientific debate about the nature of this knowledge and how it changes with experience and development make this a vibrant research area. The empirical research and theoretical debate also have implications for educational policy issues (Hirsch, 1996).

The chapter begins with a brief history of the more than 100 years of research and educational debate related to children's mathematics. The focus, however, is on contemporary research on the number, counting, and arithmetical competencies that emerge during infancy and the preschool years; elementary-school children's conceptual knowledge of arithmetic and how they solve formal arithmetic problems (e.g.,  $6 + 9$ ); and how adolescents understand and solve complex word problems. Where possible, I provide discussion of the mechanisms that support knowledge representation, problem solving, and corresponding developmental change. In the final section of the chapter, I discuss the potential mechanisms that may contribute to developmental and experience-based change in mathematical domains across a continuum, ranging from inherent, evolved abilities to abilities that are culturally specific. The discussion is meant to provide a broad outline for future scientific research on children's mathematical understanding and for thinking about how mathematics might be taught in school.

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## HISTORY

The study of arithmetical and mathematical competencies and their development has occupied psychologists and educators for more than 100 years. The approaches include experimental studies of learning, child-centered approaches to early education (constructivism), psychometric studies of individual differences in test performance, and more recently the neonativist perspective.

### Early Learning Theory

Experimental psychologists have been studying children's understanding and learning of number, arithmetic, and mathematics since the early decades of the 20th century (e.g., Brownell, 1928; O'Shea, 1901; Starch, 1911; Thorndike, 1922). The topics these researchers explored were similar to those being studied today, including the speed and accuracy with which children apprehend the quantity of sets of objects (Brownell, 1928); the strategies children use to solve arithmetic problems (Brownell, 1928); the relative difficulty of different problems (Washburne & Vogel, 1928); and the factors that influence the learning of algebra (Taylor, 1918) and geometry (Metzler, 1912). A common theme was the study of the effects of practice on the acquisition of specific mathematical competencies (Hahn & Thorndike, 1914), the factors that influence the effectiveness of practice (Thorndike, 1922), and the extent to which these competencies transferred to other domains (Starch, 1911). Practice distributed across time results in substantive improvement in the specific domain that is practiced and sometimes results in transfer to related domains. Winch (1910) found practice-related improvements in basic arithmetic were, under some conditions (e.g., student ability), associated with improved accuracy in solving arithmetical reasoning problems (multistep word problems) that involved basic arithmetic. It was also found that transfer did not typically occur across unrelated domains (Thorndike & Woodworth, 1901).

### Psychometric Studies

Psychometric studies focus on individual differences in performance on paper-and-pencil ability tests and are useful for determining whether different types of cognitive abilities exist (Thurstone, 1938), and for making inferences about the source of individual differences

(Spearman, 1927). Numerical, arithmetical, and mathematical tests have been included in these studies for more than 100 years (Cattell, 1890; Spearman, 1904), and they continue to be used to this day (Carroll, 1993). One of the most useful techniques is factor analysis, which allows correlations among ability tests to be grouped into clusters that reflect common sources of individual differences. The use of factor analysis has consistently revealed two mathematical domains, numerical facility and mathematical reasoning (e.g., Chein, 1939; Coombs, 1941; Dye & Very, 1968; French, 1951; Thurstone & Thurstone, 1941). Other relatively distinct quantitative skills, such as estimation, were also identified in some but not all psychometric studies (Canisia, 1962). Across studies, the Numerical Facility factor is defined by arithmetic computation tests (e.g., tests that require the solving of complex multiplication problems), and by tests that involve a conceptual understanding of number relationships and arithmetical concepts (Thurstone, 1938; Thurstone & Thurstone, 1941). Basically, the Numerical Facility factor appears to encompass most of the basic arithmetic skills described in the "Arithmetic in School" section. Tests that define the Mathematical Reasoning factor typically require the ability to find and evaluate quantitative relationships, and to draw conclusions based on quantitative information (French, 1951; Goodman, 1943; Thurstone, 1938).

Developmentally, Osborne and Lindsey (1967) found a distinct Numerical Facility factor for a sample of kindergarten children. The factor was defined by tests that encompassed counting, simple arithmetic, working memory for numbers, and general knowledge about quantitative relationships. Meyers and Dingman (1960) also argued that relatively distinct numerical skills are identifiable by 5 to 7 years of age. The most important developmental change in numerical facility is that it becomes more distinctly arithmetical with schooling. Mathematical reasoning ability is related to numerical facility in the elementary and middle-school years, and gradually emerges as a distinct ability factor with continued schooling (Dye & Very, 1968; Thurstone & Thurstone, 1941; Very, 1967). The pattern suggests mathematical reasoning emerges from the skills represented by the Numerical Facility factor, general reasoning abilities, and schooling.

### Constructivism

The constructivist approach to children's learning and understanding of mathematics also has a long history

(McLellan & Dewey, 1895), and continues to be influential (Ginsburg, Klein, & Starkey, 1998). There is variation in the details of this approach from one theorist to the next, but the common theme is that children's learning should be self-directed and emerge from their interactions with the physical world. McLellan and Dewey argued that children's learning of number and later of arithmetic should emerge through their manipulation of objects (e.g., grouping sets of objects). They argued their approach was better suited for children than teacher-directed instruction, and better than the emphasis on practice espoused by learning theorists (Thorndike, 1922). The most influential constructivist theory is that of Jean Piaget. Although his focus was on understanding general cognitive mechanisms (Piaget, 1950)—those applied to all domains such as number, mass, and volume—Piaget and his colleagues conducted influential studies of how children understand number (Piaget, 1965) and geometry (Piaget, Inhelder, & Szeminska, 1960).

For example, to assess the child's conception of number, two rows of seven marbles are presented and the child is asked which row has more. If the marbles in the two rows are aligned in a one-to-one fashion, then 4- to 5-year-old children almost always state that there are the same number of marbles in each row. If one of the rows is spread out such that the one-to-one correspondence is not obvious, children of this age almost always state that the longer row has more marbles. This type of justification led Piaget to argue that the child's understanding of quantitative equivalence was based on how the rows looked, rather than on a conceptual understanding of number. An understanding of number would require the child to state that, after the transformation, the number in each row was the same, even though they now looked different from one another. Children do not typically provide this type of justification until they are 7 or 8 years old, leading Piaget to argue that younger children do not possess a conceptual understanding of number. Mehler and Bever (1967) challenged this conclusion by modifying the way children's understanding was assessed and demonstrated that, under some circumstances, 2.5-year-olds use quantitative, rather than perceptual, information to make judgments about more than or less than (R. Gelman, 1972). With the use of increasingly sensitive assessment techniques, it now appears that preschool children and infants have a more sophisticated understanding of quantity than predicted by Piaget's theory (see "Early Quantitative Abilities" section).

### Information Processing

With the emergence of computer technologies and conceptual advances in information systems in the 1960s, reaction time (RT) reemerged as a method to study cognitive processes. Groen and Parkman (1972) and later Ashcraft and his colleagues (Ashcraft, 1982; Ashcraft & Battaglia, 1978) introduced these methods to the study of arithmetical processes (for a review, see Ashcraft, 1995). As an example, simple arithmetic problems, such as  $3 + 2 = 4$  or  $9 + 5 = 14$ , are presented on a computer monitor, and the child indicates by button push whether the presented answer is correct. The resulting RTs are then analyzed by means of regression equations, whereby statistical models representing the approaches potentially used to solve the problems, such as counting or memory retrieval, are fitted to RT patterns. If children count both addends in the problem, starting from one, then RTs will increase linearly with the sum of the problem, and the value of the raw regression weight will provide an estimate of the speed with which children count implicitly. These methods suggest adults typically retrieve the answer directly from long-term memory and young children typically count.

Siegler and his colleagues merged RT techniques with direct observation of children's problem solving (Carpenter & Moser, 1984) to produce a method that enables researchers to determine how children solve each and every problem; make inferences about the temporal dynamics of strategy execution; and make inferences about the mechanisms of strategy discovery and developmental change (Siegler, 1987; Siegler & Crowley, 1991; Siegler & Shrager, 1984). The results revealed that children use a mix of strategies to solve many types of quantitative problem (Siegler, 1996). Improvement in quantitative abilities across age and experience is conceptualized as overlapping waves. The waves represent the strategy mix, with the crest representing the most commonly used problem-solving approach. Change occurs as once dominant strategies, such as finger counting, decrease in frequency, and more efficient strategies, such as memory retrieval, increase in frequency.

The use of observational, RT, and other methods has expanded to the study of quantitative abilities in infancy (Antell & Keating, 1983; Wynn, 1992a) and, in fact, across the life span (Geary, Frensch, & Wiley, 1993). These methods have also been applied to the study of how children and adolescents solve arithmetical and

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algebraic word problems (Mayer, 1985; Riley, Greeno, & Heller, 1983), and to more applied issues such as mathematics anxiety (Ashcraft, 2002) and learning disabilities (Geary, 2004b).

### Neonativist Perspective

Scientific discussion of evolved mental traits arose in the latter half of the nineteenth century, following the discovery of the principles of natural selection (Darwin & Wallace, 1858). In 1871, Darwin (p. 55) noted, “[M]an has an instinctive tendency to speak, as we see in the babble of our young children; whilst no child has an instinctive tendency to . . . write.” The distinction between evolved and learned cognitive competencies was overshadowed for much of the twentieth century by psychometric, constructivist, and information-processing approaches to cognition. Neonativist approaches emerged in the waning decades of the twentieth century with the discovery that infants have an implicit, although not fully developed, understanding of some features of the physical, biological, and social worlds (e.g., Freedman, 1974; R. Gelman, 1990; R. Gelman & Williams, 1998; S. Gelman, 2003; Keil, 1992; Rozin, 1976; Spelke, Breinlinger, Macomber, & Jacobson, 1992). The neonativist emphasis on modularized competencies (e.g., language) distinguishes the approach from the general cognitive structures emphasized by Piaget.

A neonativist perspective on children’s knowledge of quantity was stimulated by R. Gelman’s and Gallistel’s (1978) *The Child’s Understanding of Number*. The gist is that children’s knowledge of counting is captured by a set of implicit principles that guide their early counting behavior and that provide the skeletal frame for learning about counting and quantity. Later, Wynn (1992a) discovered that infants have an implicit understanding of the effects of adding or subtracting one item from a small set of items, and others proposed that children’s inherent understanding of quantity could be understood in terms of a system of number-counting-arithmetic knowledge (Geary, 1994, 1995; R. Gelman, 1990; Spelke, 2000). The system was thought to be limited to an implicit ability to organize objects into sets of four or less; an ability to manipulate these sets by means of counting; and an implicit understanding of the effects of adding or subtracting from these sets. This inherent or biologically primary system was contrasted with culturally specific or biologically secondary arithmetical and

mathematical knowledge (e.g., the Base-10 system) that was learned as a result of formal or informal instruction. Although the utility of the neonativist approach is debated (Newcombe, 2002), it continues to guide theoretical and empirical research on mathematical knowledge, its development, and cross-cultural expression (Butterworth, 1999; Dehaene, Spelke, Pinel, Stanescu, & Tsivkin, 1999; Gordon, 2004; Pica, Lemer, Izard, & Dehaene, 2004).

## RESEARCH

My review of empirical and theoretical research is organized in terms of early quantitative abilities (number, counting and arithmetic before formal schooling); arithmetic learned in school; and mechanisms of change.

### Early Quantitative Abilities

My review of the quantitative abilities that emerge before formal schooling is focused on infants’ understanding of number and arithmetic; preschoolers’ understanding of number and counting; and preschoolers’ emerging arithmetic skills and conceptual knowledge of the properties of arithmetic.

#### *Number and Arithmetic in Infancy*

The study of infants’ numerical prowess has focused on three specific competencies. The first concerns the infant’s understanding of numerosity, that is, the ability to discriminate arrays of objects based on the quantity of presented items. The second and third respective competencies concern an awareness of ordinality, for example, that three items are more than two items, and the infants’ awareness of the effects of adding and subtracting small amounts from a set. I provide an overview of research on each of these competencies and then discuss underlying mechanisms.

**Numerosity.** Starkey and R. Cooper (1980) conducted one of the first contemporary investigations of the numerical competencies of infants. The procedure involved the presentation of an array of 2 to 6 dots. When an array of say three dots is first presented, the information is novel and thus the infant will orient toward the array, that is, look at it. With repeated presentations, the time spent looking at the array declines, or habituates. If the infant again orients to the array when the number of presented dots changes (dishabituation),

then it is assumed that the infant discriminated the two quantities. However, if the infant's viewing time does not change, then it is assumed the infant cannot discriminate the two quantities. With the use of this procedure, Starkey and R. Cooper found that infants between the ages of 4 and 7.5 months discriminated two items from three items but not four items from six items. Working independently, Strauss and Curtis (1984) found that 10- to 12-month-old infants discriminated two items from three items, and some of them discriminated three items from four items.

The infants' sensitivity to the numerosity of arrays of one to three, and sometimes four, items has been replicated many times and under various conditions, such as homogeneous versus heterogeneous collections of objects (Antell & Keating, 1983; Starkey, 1992; Starkey, Spelke, & Gelman, 1983, 1990; van Loosbroek & Smitsman, 1990). Among the most notable of these findings is that infants show a sensitivity to differences in the numerosity of small sets in the 1st week of life (Antell & Keating, 1983), with displays in motion (van Loosbroek & Smitsman, 1990), and intermodally (Starkey et al., 1990). In the first intermodal study, Starkey et al. presented 7-month-old infants with two photographs, one consisting of two items and the other consisting of three items, and simultaneously presented either two or three drumbeats. Infants looked longer at the photograph with the number of items that matched the number of drumbeats, suggesting that the infants somehow extracted quantity from the visual information (the photographs) and matched this with the quantity extracted from the auditory information (the drumbeats). This result suggested infants have a cognitive system that abstractly codes for numerosities up to three or four items that is neither visually nor auditorily based.

In contrast with the matching of visual and auditory quantity found by Starkey et al. (1990), infants in later studies looked at visual displays that contained a number of objects that mismatched the number of sounds or showed no preference either way (Mix, Levine, & Huttenlocher, 1997; Moore, Benenson, Reznick, Peterson, & Kagan, 1987). The former results might be interpreted as evidence for an ability to discriminate a small number of objects from a sequence of sounds, but the mixed pattern of results leaves the issue unresolved. Other investigators have questioned whether infants' numerosity discriminations are based on an implicit understanding of quantity or other factors that are correlated with quantity in the experimental procedures

(Clearfield & Mix, 1999; Newcombe, 2002; Simon, 1997). As an example, as the number of items increases, so does total surface area, overall duration of tones, and so forth. Clearfield and Mix confirmed that infants can discriminate small sets of squares that differ in quantity, but the discriminations are more strongly influenced by the length of the perimeter of the squares than by number of squares (see also Feigenson, Carey, & Spelke, 2002).

To control for some of these perceptual-spatial and other confounds, Wynn and Sharon (Sharon & Wynn, 1998; Wynn, 1996) presented 6-month-old infants with a puppet that engaged in a series of two or three actions (see also Wynn, Bloom, & Chiang, 2002). In a two-action sequence, an infant might first see the puppet jump, stop briefly, and then jump again. Evidence for infants' ability to enumerate small numbers of items was provided by their ability to discriminate between two- and three-action sequences. In a series of related studies, Spelke and her colleagues found that 6-month-old infants can discriminate visually and auditorily presented sets of larger quantities from smaller quantities, but only if the number of items in the larger set is double that in the smaller set (e.g., 16 versus 8; Brannon, Abbott, & Lutz, 2004; Lipton & Spelke, 2003; Xu & Spelke, 2000). The results were interpreted as supporting the hypothesis that people, including infants, have an intuitive sense of approximate magnitudes (Dehaene, 1997; Gallistel & Gelman, 1992), but the mechanisms underlying infants' performance in Spelke's studies may differ from those underlying infants' discrimination of sets of two items, sounds, or actions, from sets of three items, sounds, or actions, as described in the "Representational Mechanisms" section.

**Ordinality.** Even though infants are able to detect and represent small quantities, this does not mean they are necessarily sensitive to which set has *more* or *less* items. Initial research suggested that infants' sensitivity to these ordinal relationships, that is somehow knowing that 3 is more than 2 and 2 is more than 1, was evident by 18 months (R. Cooper, 1984; Sophian & Adams, 1987; Strauss & Curtis, 1984), although more recent studies suggest a sensitivity to ordinality may emerge by 10 months (Brannon, 2002; Feigenson, Carey, & Hauser, 2002). In an early study, Strauss and Curtis taught infants, by means of operant conditioning, to touch the side of a panel that contained the smaller or larger number of two arrays of dots. For the smaller-reward condition, the

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smaller array might contain three dots and the larger array four dots; infants would then be rewarded for touching the panel associated with the smaller array. The infants might next be presented with arrays of two and three dots. If infants were simply responding to the rewarded value, they should touch the panel associated with three. In contrast, if infants respond based on the ordinal relationship (responding to the smaller array in this example), then they would touch the panel associated with two. In this study, 16-month-olds responded to the two, suggesting a sensitivity to less than (Strauss & Curtis, 1984). Ten- to 12-month-olds seemed to notice that the numerosities in the arrays had changed, but they did not discriminate less than and more than; it was simply different than (R. Cooper, 1984).

Feigenson, Carey, and Hauser (2002) adopted a procedure used to study rhesus monkeys' (*Macaca mulatta*) sensitivity to ordinality (Hauser, Carey, & Hauser, 2000). In their first experiment, crackers of various size and number were placed, one at a time, in two opaque containers and the infant was allowed to crawl to the container of their preference. Groups of 10- and 12-month-old infants consistently chose the container with the larger number of crackers when the comparison involved 1 versus 2 and 2 versus 3. When four or more crackers were placed in one of the containers, performance dropped to chance levels. Another experiment pitted number against total amount (i.e., surface area) of cracker. In this experiment, 3 out of 4 infants chose the container with the greater amount of cracker (1 large cracker) as opposed to the container with the greater number of crackers (2 smaller crackers). Brannon (2002) demonstrated that 11-month-old, but not 9-month-old, infants could discriminate ordinally arranged sets that increased (e.g., 2, 4, 8 items) or decreased (e.g., 8, 4, 2 items) in quantity. A variety of experimental controls suggested that the discriminations were based on the change in number of items in the sets, and not change in surface area or perimeter.

**Arithmetic.** Wynn (1992a) provided the first evidence that infants may implicitly understand simple arithmetic. In one experiment, 5-month-olds were shown a Mickey Mouse doll in a display area. A screen was then raised and blocked the infant's view of the doll. Next, the infant watched the experimenter place a second doll behind the screen. The screen was lowered and showed either one or two dolls. Infants tend to look longer at unexpected events (but see Cohen & Marks,

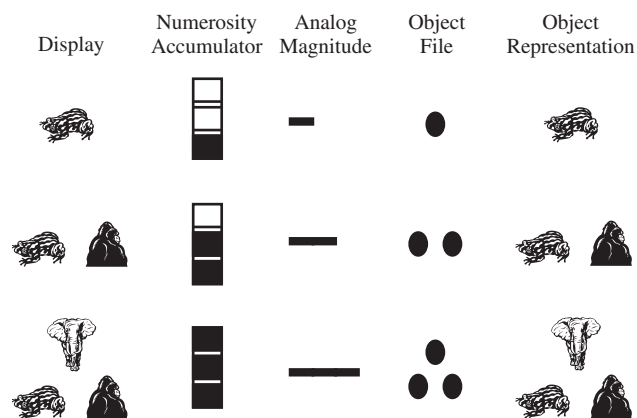
2002). Thus, if infants are aware that adding a doll to the original doll will result in more dolls, then, when the screen is lowered, they should look longer at one doll than at two doll displays. This is exactly the pattern that was found. Another procedure involved subtracting one doll from a set of two dolls and, again, infants looked longer at the unexpected (two dolls) than expected (one doll) outcome. The combination of results was interpreted as evidence that infants have an implicit understanding of imprecise addition and subtraction; they understand addition increases quantity and subtraction decreases quantity. Another experiment suggested infants may also have an implicit understanding of "precise" addition; specifically, that one item + one item = exactly two items.

Wynn's (1992a) empirical results and the claim they provide evidence for arithmetical competencies in infancy have been the focus of intense scientific study and theoretical debate (Carey, 2002; Cohen, 2002; Cohen & Marks, 2002; Kobayashi, Hiraki, Mugitani, & Hasegawa, 2004; Koechlin, Dehaene, & Mehler, 1997; McCrink & Wynn, 2004; Simon, Hespos, & Rochat, 1995; Uller, Carey, Huntley-Fenner, & Klatt 1999; Wakeley, Rivera, & Langer, 2000; Wynn, 1995, 2000, 2002; Wynn & Chiang, 1998). The findings for imprecise addition and (or) subtraction have been replicated several times (Cohen & Marks, 2002; Simon et al., 1995; Wynn & Chiang, 1998). In an innovative cross-modality assessment, Kobayashi et al., demonstrated that 5-month-olds can add one object and one tone or one object and two tones. In contrast, Koechlin et al., replicated Wynn's finding for imprecise subtraction but not addition, and Wakeley et al. failed to replicate the findings for imprecise or precise addition or subtraction. Wakeley et al. suggested that if infants do have implicit knowledge of arithmetic, then it is expressed only under some conditions. Other scientists have suggested that the phenomenon, that is, infants' looking patterns described in Wynn's original study, is robust but due to factors other than arithmetic competency (Cohen & Marks, 2002; Simon et al., 1995).

**Representational Mechanisms.** The proposed perceptual and cognitive mechanisms that underlie infants' performance on the tasks previously described range from a system that encodes and can operate on abstract representations of numerosity (Starkey et al., 1990; Wynn, 1995) to mechanisms that result in performance that appears to be based on an implicit understanding of numerosity but in fact tap systems that are

designed for other purposes. Included among these alternatives are mechanisms that function for object identification and individuation (Simon et al., 1995), and mechanisms that are sensitive to the contour and length of the presented objects (Clearfield & Mix, 1999; Feigenson, Carey, & Spelke, 2002). Cohen and Marks (2002) argued that an attentional preference for an optimal mix of familiarity and complexity explained Wynn's (1992a) findings. To illustrate, in Wynn's original  $1 + 1 = 2$  manipulation, the expected result is two dolls and the unexpected result is one doll. However, the unexpected result is also the original condition—one doll—viewed by the infants, and thus their looking preference might have been based on familiarity, not an expectation of two dolls. Kobayashi et al.'s (2004) cross-modality study controlled for familiarity and still provided evidence that infants can add small quantities, an object and one or two tones.

Figure 18.1 provides an illustration of the mechanisms that have been proposed as potentially underlying infants' performance on quantitative tasks. On the basis of Meck's and Church's (1983) studies of the numerosity and time-estimation skills in animals, Gallistel and Gelman (1992) proposed the existence of an innate preverbal counting mechanism in humans. There are two ways in which preverbal counting might operate. The



**Figure 18.1** Potential mechanisms underlying infants' and young children's implicit understanding of the number. The numerosity accumulator exactly represents the number of one to three or four objects. The analog magnitude mechanism represents quantity, including number and area, but inexactly. The object file mechanism differentiates from one to three or four objects represented in visual short-term memory, and only incidentally provides information on number. The object representation mechanism is the same as the object file but in addition represents features of each object (e.g., color, shape).

first is represented by the numerosity accumulator. The gist is that infants have a mechanism that accumulates representations of up to three or four objects, sounds, or events and then compares the accumulated representation (e.g., one, or two) to abstract and inherent knowledge of the numerosity of one, two, three, and perhaps four items. The second is an analog mechanism that can represent various types of magnitudes (e.g., surface area), including numerosities of any size, but with increases in magnitude size the precision of the estimate decreases. Object file and object representation mechanisms (e.g., Kahneman, Treisman, & Gibbs, 1992) function to individuate and represent whole objects rather than numerosity. The object file generates a marker for each object in the display, whereas the object representation mechanism adds object-specific information (e.g., area, contours) to each object marker. The number of object markers that can be simultaneously held in short-term memory is three or four (Trick, 1992). Thus, these object individuation mechanisms incidentally provide numerical information.

Starkey et al. (1990), Wynn (1992a, 1995), and Gallistel and Gelman (1992), among others (e.g., Spelke, 2000), have interpreted the numerosity, ordinality, and arithmetic performance of infants as being consistent with the existence of a numerosity accumulator and an analog magnitude mechanism that has evolved specifically to represent quantity. Simon et al. (1995) and Newcombe (2002), among others (e.g., Uller et al., 1999), have argued that the same pattern of results can be explained by the object file or object representation mechanisms. A set of one item can be discriminated from a set of two items because object files representing the two sets look different, not because of an inherent understanding of the "oneness" and "twoness" of the sets. It is also possible the object file systems evolved for individuating whole objects, and that numerosity mechanisms evolved later but are dependent on information generated by the object file mechanisms; specifically, the object file information might be fed into the numerosity accumulator. The best evidence to date for an independent numerosity accumulator comes from Kobayashi et al.'s (2004) cross-modality results, given that the tones were not likely to have been represented by object file mechanisms.

At this point, a definitive conclusion cannot be drawn regarding these alternative mechanisms. Whatever mechanisms may be operating, it is clear that infants can make discriminations based on the numerosity of sets of

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three to four items (Starkey et al., 1990) and can later make simple ordinality judgments (Straus & Curtis, 1984), although their understanding of simple arithmetic is less certain. Spelke's recent studies suggest that infants can also make discriminations among larger sets of items, but only when the sets differ by a magnitude of 2 to 1 (e.g., 16 versus 8; Lipton & Spelke, 2003; Xu & Spelke, 2000). These findings support the existence of an inherent mechanism that represents analog magnitudes, but it is not known if the evolved function of the system is to represent numerosity per se or other forms of magnitude, such as area or distance. A recent neuroimaging study suggests that the brain regions that represent numerical magnitude also represent spatial magnitude (related size of objects) and thus these regions may not be specifically numerical in function (Pinel, Piazza, Le Bihan, & Dehaene, 2004).

### *Number and Counting during Preschool*

Counting and number-related activities are common in industrialized and traditional societies (Crump, 1990), although the extent and form of formal number-counting systems (e.g., number words) varies considerably from one culture to the next (Gordon, 2004). Saxe studied the representational and counting systems of the Oksapmin, a horticultural society in Papua New Guinea (Saxe, 1981, 1982). Here, counting and numerical representations are mapped onto 27 body parts. "To count as Oksapmins do, one begins with the thumb on one hand and enumerates 27 places around the upper periphery of the body, ending on the little finger of the opposite hand" (Saxe, 1982, pp. 159–160). This system is used not only for counting, but also for representing ordinal position and for making basic measurements. In many cultures, there are common activities, such as parent-child play, during the preschool years and childhood that facilitate children's understanding of counting and number, and their learning the specific representational systems (e.g., number words) of the culture (Saxe, Guberman, & Gearhart, 1987; Zaslavsky, 1973).

Although these patterns suggest that children's interest in understanding the rudiments of counting and number may be inherent (Geary, 1995; R. Gelman & Gallistel, 1978), achieving a mature understanding of these concepts and their use in culturally appropriate ways spans extends into early childhood (Fuson, 1988; Piaget, 1965). In the following sections, I provide an overview of how children's understanding of number and counting skills change during the preschool years.

These changes include the acquisition of number concepts, number words, counting procedures, and an understanding of the use of numbers and counting for representing cardinality, ordinality, and for making measurements.

**Number Concepts.** To use number knowledge explicitly (e.g., to count) and to extend number concepts beyond small quantities, children must map their culture's number words and other representational systems (e.g., Arabic numerals) onto the numerosity and analog magnitude systems (Spelke, 2000). The process begins sometime between 2 and 3 years when children begin to use number words during the act of counting (R. Gelman & Gallistel, 1978), although young children do not always use the standard order of word tags (one, two, three). The child might state "three, five" to count two items, and "three, five, six" to count three items. Despite these apparent errors, this pattern has two important aspects; each number word is used only once during each count, and the sequence is stable across counted sets. For children who show this type of pattern, the implication is they implicitly understand that different number words represent different quantities and that the sequence with which the words are stated is important (R. Gelman & Gallistel, 1978). Many children as young as 2.5 years also understand that number words are different from other descriptive words. When children are asked to count a row of three red toy soldiers, they often use number words to count the set. The children understand that red describes an attribute of the counted items, but the number assigned to each soldier does not describe an attribute of the soldier but somehow refers to the collection of soldiers (Markman, 1979).

R. Gelman and Gallistel (1978; Gallistel & Gelman, 1992) argued regularities in the use of counting and number words emerge because children implicitly and automatically map counting onto representations of quantity, although the mechanisms that support this mapping are not fully understood (Sarnecka & Gelman, 2004). In any case, most 2-year-olds have not yet mapped specific quantities onto specific number words (Wynn, 1992b). Many 2.5-year-olds can discriminate four-item sets from three-item sets, and often know that the Arabic numeral 4 represents more than the numeral 3 (Bullock & Gelman, 1977), but might not be able to correctly label the sets as containing "four" and "three" items, respectively. Wynn argued that it re-

quires as long as a year of counting experience, from 2 to 3 years of age, for children to begin to associate specific number words with their mental representations of specific quantities and then to use this knowledge in counting tasks. It is likely that many 3-year-olds are beginning to associate specific number words with specific quantities, but this knowledge appears to extend only to the quantities that can be represented by the numerosity accumulator or object file systems.

The learning of specific quantities beyond four and mapping number words onto the representations of these quantities is a difficult task, because the analog magnitude system that must be adapted for this purpose functions to represent general amounts, not specific quantities (Gallistel & Gelman, 1992). Children's conception of the quantities represented by larger numbers is thus dependent to an important degree on their learning of the standard counting sequence (1, 2, 3, . . .) and properties of this sequence (e.g., successive number words represent an increase of exactly one). The conception of number also appears to be related to an ability to generate a mental number line (Dehaene, 1997), which involves mapping the Arabic number sequence onto the analog magnitude system. However, the ability to use the mental number line to represent specific quantities only emerges with formal schooling (Siegler & Opfer, 2003), as described in the "Arithmetic in School" section.

**Number Words.** Learning the culture's number words is an essential step in children's mathematical development. As noted, knowledge of the sequence of number words enables children to develop more precise representations of numbers greater than three or four, and contributes to knowledge of cardinality and ordinality described in the next section (Fuson, 1988; R. Gelman & Gallistel, 1978; Wynn, 1990). Most 3- to 4-year-olds know the correct sequence of number words from one to ten (Fuson, 1988; Siegler & Robinson, 1982). For children speaking most European-derived languages, including English, learning number words greater than ten is difficult (Fuson & Kwon, 1991; Miller & Stigler, 1987). This is because the number words for values up to the hundreds are often irregular; they do not map onto the underlying base-10 structure of the number system. For example, *twelve* is another number word in a continuous string of number words. Its special status—repetition of basic unit values—within the base-10 system is not evident from the

word itself. These irregularities slow the learning of number words and the understanding of the corresponding quantity the word represents, and result in word tagging errors, such as writing 51 when hearing 15 (Ginsburg, 1977).

Many of these confusions are avoided in East Asian languages, because of a direct one-to-one relation between number words greater than ten and the underlying base-10 values represented by the words (Fuson & Kwon, 1991; I. Miura, Kim, Chang, & Okamoto, 1988; I. Miura, Okamoto, Kim, Steere, & Fayol, 1993). The Chinese word for twelve is translated as "ten two." Using ten two to represent 12 has two advantages. First, children do not need to memorize additional word tags, such as eleven and twelve. Second, the fact that twelve is composed of a single tens value and two units values is obvious. As might be expected, the irregular structure of European-derived number words is not associated with cross-national differences in children's learning of number words less than 10, their ability to solve simple arithmetical word problems that involve manipulating quantities less than 10, or their understanding of the counting principles to be described (Miller, Smith, Zhu, & Zhang, 1995). However, differences in the structure of number words between 10 and 100 appear to contribute to later cross-national differences in ease of learning the base-10 structure of the number system and to use associated arithmetical procedures and problem-solving strategies (Geary, Bow-Thomas, Liu, & Siegler, 1996; Fuson & Kwon, 1992a).

**Counting Procedures and Errors.** The conceptual principles that guide counting behavior emerge during the preschool years, but it is debated as to whether these principles have an inherent basis (R. Gelman & Gallistel, 1978), or emerge through observation of the counting behavior of others and regularities in this behavior (Briars & Siegler, 1984; Fuson, 1988). And, of course, counting principles may emerge from a combination of inherent constraints and counting experience (Geary, 1995; R. Gelman, 1990). R. Gelman and Gallistel proposed children's counting behavior is guided by five inherent and implicit principles that mature during the preschool years. During those years, children's counting behavior and their description of counting suggest knowledge of these implicit rules can become more explicit and the application of these principles during the act of counting becomes more stable and accurate.

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The principles are one-one correspondence (one and only one word tag, e.g., “one,” “two,” is assigned to each counted object); stable order (the order of the word tags must be invariant across counted sets); cardinality (the value of the final word tag represents the quantity of items in the counted set); abstraction (objects of any kind can be collected together and counted); and order-irrelevance (items within a given set can be tagged in any sequence). The principles of one-one correspondence, stable order, and cardinality define the initial “how to count” rules, which, in turn, provide the skeletal structure for children’s emerging knowledge of counting (R. Gelman & Meck, 1983).

Children make inductions about the basic characteristics of counting by observing standard counting behavior and associated outcomes (Briars & Siegler, 1984; Fuson, 1988). These inductions may elaborate and add to R. Gelman and Gallistel’s (1978) counting rules. One result is a belief that certain unessential features of counting are essential. These unessential features include standard direction (counting must start at one of the end points of a set of objects), and adjacency. The latter is the incorrect belief that items must be counted consecutively and from one contiguous item to the next; “jumping around” during the act of counting results in an incorrect count. By 5 years of age, many children know most of the essential features of counting described by R. Gelman and Gallistel but also believe that adjacency and start at an end are essential features of counting. The latter beliefs indicate that young children’s understanding of counting is influenced by observation of standard counting procedures and is not fully mature.

An implicit conceptual understanding of counting does not mean that children will not make counting errors; they do. Fuson (1988) has extensively documented the many different types of error committed by children while they count. A common error involves the child pointing to each item once, but speaking two or more word tags with each point. Sometimes children correctly assign one word tag to each counted object, but speak additional word tags, without pointing, in between counted objects. Sometimes children point and tag items with each syllable of a number word, and at other times they might tag and point to each item two or more times. Despite all the different types of errors that can be made during the act of counting, kindergarten children are typically proficient counters, especially for smaller set sizes.

**Cardinality and Ordinality.** Although an implicit understanding of cardinality and ordinality appears to emerge before children learn the sequence of number words (Bermejo, 1996; Brainerd, 1979; Brannon & Van de Walle, 2001; R. Cooper, 1984; Huntley-Fenner & Cannon, 2000; Ta’ir, Brezner, & Ariel, 1997; Wynn, 1990, 1992b), the mapping of these concepts onto the counting sequence is an essential step in the development of arithmetical competencies. Children must learn that the number word assigned to the last counted object can be used to represent the total number of counted objects—*cardinality*—and that successive number words represent successively larger quantities—*ordinality*.

One way to assess children’s understanding of cardinality is to ask them to count their fingers and then ask “how many fingers do you have?” Children who do not understand the significance of the last word tag, *five* in this example, will recount their fingers, rather than restating “five” (Fuson, 1991). Although some 3- and many 4- and 5-year-olds perform well on such cardinality tasks suggesting a developing sense of the concept, most preschoolers’ understanding of cardinality is not yet mature. Their performance is inconsistent and often influenced by factors other than cardinality. For small set sizes (e.g., less than 10), most children have a good grasp of cardinal value by 5 years (Bermejo, 1996; Freeman, Antonucci, & Lewis, 2000), but generalization to larger sets and a consistent focus on cardinality information over other information (e.g., perceptual cues) may not emerge for several more years (Piaget, 1965).

As with infants, the demonstration of preschool children’s knowledge of ordinal relationships requires special techniques. Bullock and Gelman (1977) assessed the ordinal knowledge of 2- to 5-year-old children, by using a “magic” game. In the first of two phases, the children were shown two plates of toys, one contained a single toy animal and the other contained two toy animals. The children were either told that the two-toy plate was the winner (more condition) or that the one-toy plate was the winner (less condition). The child was then shown a series of one- and two-toy plates, and asked to pick the winner. In Phase II, “the experimenter surreptitiously added one animal to the two-toy plate and three animals to the one-toy plate” (p. 429). The critical question was whether the children would choose the winner based on the relation, that is, more or less, that was reinforced in the first phase. The majority of 3- and 4-year-olds responded based on the relational information, but less than 50% of the 2-year-olds did. When

the memory demands of the task were reduced, more than 90% of the 2 year olds responded based on the relational information. More recent studies suggest that some 2-year-olds can make ordinal judgments comparing sets as high as five objects versus six objects, independent of perceptual factors (e.g., total surface area of the objects) or verbal counting (Brannon & Van de Walle, 2001; Huntley-Fenner & Cannon, 2000).

These results suggest children as young as 2 years may have an understanding of ordinal relationships that extend to larger set sizes than found in the infancy studies. The mechanisms underlying this knowledge are not yet known, but Huntley-Fenner and Cannon (2000) argued that the analog magnitude mechanism shown in Figure 18.1 is the source of preschool children's ordinality judgments. This is a reasonable conclusion, given that knowledge of ordinal relations for larger numbers is based on knowledge of the counting-word sequence and that this sequence is likely mapped onto the analog system (Gallistel & Gelman, 1992).

### *Arithmetic during Preschool*

As I overview in the first section, research on preschooler's emerging arithmetic skills has focused on their implicit understanding of the effects of addition and subtraction on number; their implicit understanding of some properties of arithmetic (e.g., that addition and subtraction are inverse operations); and their integration of counting and arithmetic knowledge to develop procedures for solving formal arithmetic problems (e.g.,  $3 + 2 = ?$ ). As I note in the second section, there is vigorous debate regarding the relation between these emerging skills and the representational mechanisms that were described in Figure 18.1.

**Arithmetical Competencies.** Some of the earliest evidence of preschoolers' implicit knowledge of arithmetic was provided by Starkey (1992; see also Sophian & Adams, 1987; Starkey & Gelman, 1982). Here, a searchbox task was designed to determine if 1.5- to 4-year-olds understand how addition and subtraction affects numerosity without using verbal counting. The searchbox "consisted of a lidded box with an opening in its top, a false floor, and a hidden trap-door in its back. This opening in the top of the box was covered by pieces of elastic fabric such that a person's hand could be inserted through pieces of the fabric and into the chamber of the searchbox without visually revealing the chamber's contents" (Starkey, 1992, p. 102). In the first of

two experiments, children placed between one and five table-tennis balls, one at a time, into the searchbox. Immediately after the child placed the last ball, she was told to take out all the balls. An assistant had removed the balls and then replaced them one at a time, as the child searched for the balls. If a child placed three balls in the searchbox and then stopped searching once three balls were removed, then it could be argued that this child was able to represent and remember the number of balls deposited, and used this representation to guide the search. The results showed that 2-year-olds can mentally represent and remember numerosities of one, two, and sometimes three; 2.5-year-olds can represent and remember numerosities up to and including three; and 3- to 3.5-year-olds can sometimes represent numerosities as high as four.

The second experiment addressed whether the children could add or subtract from these quantities without verbally counting. The same general procedure was followed, except that once all the original balls were placed in the searchbox, the experimenter placed an additional 1 to 3 balls in the box, or removed from 1 to 3 balls. If children understand the effect of addition, then they should search for more balls than they originally placed in the searchbox, or for subtraction they should stop searching before this point. Nearly all 2- to 4-year-olds responded in this manner, as did many, but not all, of the 1.5-year-olds. Examination of the accuracy data indicated that many of the 1.5-year-olds were accurate for addition and subtraction with sums or minuends (the first number in a subtraction problem) less than or equal to two (e.g.,  $1 + 1$  or  $2 - 1$ ), but were not accurate for problems with larger numbers. Most 2-year-olds were accurate with values up to and including three (e.g.,  $2 + 1$ ), and there was no indication (e.g., vocalizing number words) they used verbal counting to solve the problems. None of the children were accurate with sums or minuends of four or five.

Using a different form of nonverbal calculation task, Huttenlocher, Jordan, and Levine (1994) found that 2- to 2.5-year-olds understand that addition increases quantity and subtraction decreases it, but only a small percentage of them could solve simple 1 object + 1 object or 2 objects - 1 object problems. The ability to calculate exact answers improved for 2.5- to 3-year-olds such that more than 50% of them could mentally perform simple additions ( $1 + 1$ ) and subtractions ( $2 - 1$ ) of objects. Some 4-year-olds correctly solved nonverbal problems that involved 4 objects + 1 object, or 4 objects - 3 objects.

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Klein and Bisanz (2000) used the same procedure to assess 4-year-olds but also administered three-step problems (e.g., 2 objects + 1 object – 1 object) and assessed potential predictors of problem difficulty as measured by error rate. The three-step problems assessed children's understanding of the inverse relation between addition and subtraction; 5 objects – 1 object + 1 object = 5 objects, because 1 object – 1 object results in no change in quantity. The indices of problem difficulty included the value of numbers in the problem, the answer, and the largest of these numbers, which was termed representational set size. So, for 5 objects + 1 object – 1 object, the representational set size is five, whereas for 3 + 1, the size is four. Representational set size provided an index of the working memory demands of the problem and thus provided an estimate of the capacity of the underlying mechanism for representing object sets (e.g., object file or numerosity accumulator).

The results revealed that error rates increased with increases in representational set size. Holding an object file or numerosity-accumulator file of four or five items in working memory and adding or subtracting from this representation was more difficult than holding an object file or numerosity-accumulator file of two or three in working memory. Thus, errors are due to a lack of ability to apply arithmetical knowledge to larger sets or (and) to working memory failures. The strategies children used to solve the inversion problems suggested that some of these children had a nascent and implicit understanding of the associative property of addition  $[(a + b) - c = a + (b - c)]$ , and the inverse relation between addition and subtraction. Knowledge of the former was inferred “when children overtly subtracted the  $c$  term from the  $a$  term and then added the  $b$  term, thus transforming  $(a + b) - c$  to  $(a - c) + b$ ” (Klein & Bisanz, 2000, p. 110).

Vilette (2002) administered a variant of Wynn's (1992a) task to 2.5-, 3.5-, and 4.5-year-olds; the problems involved  $2 + 1$ ,  $3 - 1$ , and an inversion problem,  $2 + 1 - 1$ . For these latter problems, the children first saw two dolls on a puppet theater stage. Next, a screen rotated up and the experimenter placed another doll, in full view of the child, behind the screen, and then placed his hand back behind the screen and removed a doll. The screen was then lowered and revealed two (expected) or three (unexpected) dolls. The children were instructed to respond “normal” (correct) to the expected result and “not normal” to the unexpected result. As a group, 2.5-year-olds solved 64% of the  $1 + 1$  problems, but failed

nearly all the  $3 - 1$  and inversion problems. The 4.5-year-olds solved all the problems with greater than 90% accuracy, in keeping with Klein's and Bisanz's (2000) findings that some 4-year-olds have an implicit understanding of the inverse relation between addition and subtraction. The negative findings for younger children need to be interpreted with caution, because poor performance could be due to the working memory demands of the task or to lack of arithmetical knowledge.

In any event, by 4 to 5 years, children have coordinated their counting skills, number concepts, and number words with their implicit arithmetical knowledge. The result is an ability to use number words to solve simple addition and subtraction problems (Baroody & Ginsburg, 1986; Groen & Resnick, 1977; Saxe, 1985; Siegler & Jenkins, 1989). The specific strategies used by children for solving these problems may vary somewhat from one culture to the next, but typically involve the use of concrete objects to aid the counting process. When young children in the United States are presented with verbal problems, such as “how much is  $3 + 4$ ,” they will typically represent each addend with a corresponding number of objects, and then count all the objects starting with one; counting out three blocks and then four blocks, and then counting all the blocks. If objects are not available, then fingers are often used as a substitute (Siegler & Shrager, 1984). Korean and Japanese children apparently use a similar strategy (Hatano, 1982; Song & Ginsburg, 1987). Somewhat older Oksapmin use an analogous strategy based on their body-part counting system (Saxe, 1982).

**Mechanisms.** There is debate whether the mechanisms underlying preschoolers' arithmetic competencies are inherently numerical (Butterworth, 1999; Dehaene, 1997; Gallistel & Gelman, 1992; Geary, 1995) or emerge from nonnumerical mechanisms (Huttenlocher et al., 1994; Houdé, 1997; Jordan, Huttenlocher, & Levine, 1992; Vilette, 2002). Inherent mechanisms would involve some combination of the numerosity accumulator and analog magnitude system (Spelke, 2000), with the assumption that the latter automatically results in at least imprecise representations of the quantities being processed (Dehaene, 1997; Gallistel & Gelman, 1992). Integrated with these systems for representing numerosity would be an inherent understanding of at least basic principles of addition (increases quantity) and subtraction (decreases quantity), and a later integration with counting competencies to provide a procedural system

for solving arithmetic problems. These inherent biases are thought to provide the core knowledge that initially guides children's attention to numerical (e.g., seeing a group of objects as sets) and arithmetical (e.g., combining sets of objects) aspects of their experiences. These biases interact with these experiences to flesh out their conceptual knowledge and procedural competencies (Geary, 1995; R. Gelman, 1990). The fleshing out would involve extension of this knowledge and procedures to larger and larger quantities as well as making inductions about regularities in arithmetical processes and outcomes (e.g., inverse relation between addition and subtraction). In contrast, Huttenlocher et al. argued that preschoolers' arithmetical competencies emerge from a combination of a domain-general ability to mentally represent symbols—objects in the object file (as was shown in Figure 18.1)—and general intelligence, whereby children learn to manipulate representations of these symbols based on the rules of arithmetic they induce by engaging in arithmetical activities.

### Arithmetic in School

As children move into formal schooling, the breadth and complexity of the mathematics they are expected to learn increases substantially. The task of understanding children's development in these areas has become daunting, and our understanding of this development is thus only in the beginning stages. In the first section, I make a distinction between potentially inherent and culturally-universal quantitative competencies and culturally-specific and school-dependent competencies, which I return to in "Mechanisms of Change." In the remaining sections, I provide an overview of children's conceptual knowledge of arithmetic in the elementary school years; how children solve formal arithmetic problems (e.g.,  $16 - 7$ ); and the processes involved in solving word problems.

### Primary and Secondary Competencies

Most of the competencies reviewed in the "Early Quantitative Abilities" section emerge with little if any instruction, and some may be in place in the first days or weeks of life. As far as we know, most of these competencies are found across human cultures, and many are evident in other species of primate (Beran & Beran, 2004; Boysen & Berntson, 1989; Brannon & Terrace, 1998; Gordon, 2004; Hauser et al., 2000) and in at least

a few nonmammalian species (Lyon, 2003; Pepperberg, 1987). The combination of early emergence, cross-cultural ubiquity, and similar competencies in other species is consistent with the position that some of these competencies are inherent (Geary, 1995, R. Gelman, 1990; R. Gelman & Gallistel, 1978), no matter whether the evolved function is specifically quantitative. In contrast, much of the research on older children's development is focused on school-taught mathematics. It is thus useful to distinguish between inherent and school-taught competencies, because it is likely that some of the mechanisms involved in their acquisition differ, as discussed in the "Mechanisms of Change" section. Geary (1994, 1995; see also Rozin, 1976) referred to inherent forms of cognition, such as language and early quantitative competencies, as *biologically primary abilities*, and skills that build on these abilities but are principally cultural inventions, such as base-10 arithmetic, as *biologically secondary abilities*.

### Conceptual Knowledge

**Properties of Arithmetic.** The commutativity and associativity properties of addition and multiplication have been understood by mathematicians since the time of the ancient Greeks. The commutativity property concerns the addition or multiplication of two numbers and states that the order in which the numbers are added or multiplied does not affect the sum or product ( $a + b = b + a$ ;  $a \times b = b \times a$ ). The associativity property concerns the addition or multiplication of three numbers and again states that the order in which the numbers are added or multiplied does not affect the sum or product [ $(a + b) + c = a + (b + c)$ ;  $(a \times b) \times c = a \times (b \times c)$ ]. Both commutativity and associativity involve the decomposition (e.g., of the sum) and recombination (e.g., addition) of number sets; specifically, an understanding that numbers are sets composed of smaller-valued numbers that can be manipulated in principled ways. Research on children's understanding of these properties has focused largely on addition and commutativity (Baroody, Ginsburg, & Waxman, 1983; Resnick, 1992), although some research has also been conducted on children's understanding of associativity (Canobi, Reeve, & Pattison, 1998, 2002).

One approach has been to study when and how children acquire an understanding of commutativity and associativity as related to school-taught arithmetic (Baroody et al., 1983). The other approach has focused on the more basic and potentially biologically primary

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numerical and relational knowledge that allows children to later understand commutativity in formal arithmetical contexts (R. Gelman & Gallistel, 1978; Resnick, 1992; Sophian, Harley, & Martin, 1995). The latter approach meshes with the ability of infants and preschool children to mentally represent and manipulate sets of small quantities of objects.

Resnick (1992) proposed that aspects of these competencies provide the foundational knowledge for children's emerging understanding of commutativity as an arithmetical principle. The first conceptual step toward building an explicit understanding of commutativity is prenumerical and emerges as children physically combine sets of objects. One example would involve the physical merging of sets of toy cars and sets of toy trucks, and the understanding (whether inherent or induced) that the same group of cars and trucks is obtained whether the cars are added to the trucks or the trucks are added to the cars. The next step in Resnick's model involves mapping specific quantities onto this knowledge, as in understanding that  $5 \text{ cars} + 3 \text{ trucks} = 3 \text{ trucks} + 5 \text{ cars}$ . At this step, children only understand commutativity with the addition of sets of physical objects. The third step involves replacing physical sets with numerals, as in knowing that  $5 + 3 = 3 + 5$ , and the final step is formal knowledge of commutativity as an arithmetical principle, that is, understanding  $a + b = b + a$ . Empirical studies of children's emerging knowledge of commutativity are consistent with some but not all aspects of Resnick's model (Baroody, Wilkins, & Tiihikainen, 2003).

An implicit understanding that sets of physical objects can be decomposed and recombined emerges at about 4 years of age (Canobi et al., 2002; Klein & Bisanz, 2000; Sophian & McCorgray, 1994; Sophian et al., 1995). Most children of this age understand, for instance, that the group of toy vehicles includes smaller subsets of cars and trucks. Between 4 and 5 years, an implicit understanding of commutativity as related to Resnick's (1992) second step is evident for many children (Canobi et al., 2002; Sophian et al., 1995). As an example, Canobi et al. presented children with two different colored containers that held a known number of pieces of candy. The two containers were then given in varying orders to two toy bears and the children were asked to determine if the bears had the same number of candy pieces. The task was to determine if, for example, a container of 3 candies + a container of 4 candies was equal to a container of 4 candies + a container of 3 can-

dies, and was therefore a commutativity problem. In a variant of this task, a third set of candies was added to one of the two containers. For one trial, four green candies were poured into a container of three red candies, a manipulation that is analogous to  $(4 + 3)$ . When presented with the second container of candy, the task represented a physical associativity problem; (a container of 3 candies + 4 candies) + a container of 4 candies. In keeping with Resnick's (1992) model, most of the 4-year-olds implicitly understood commutativity with these physical sets (see also Sophian et al., 1995), but many of them did not grasp the concept of associativity (Canobi et al., 2002).

Inconsistent with Resnick's (1992) model was the finding that performance on the commutativity task was not well integrated with their knowledge of addition. In other words, many of the children solved the task correctly because both bears received a *collection* of red candies and a *collection* of green candies and they understood that the order in which the bears received the containers of candy did not matter. Many of the children did not, however, approach the task as the addition of specific quantities.

Baroody et al. (1983) assessed children's understanding of commutativity as expressed during the actual solving of addition problems in a formal context (e.g.,  $3 + 4 = 4 + 3$ )—Resnick's third step—in first, second, and third graders. The children were presented with a series of formal addition problems, in the context of a game. Across some of the trials, the same addends were presented but with their positions reversed. For example, the children were asked to solve  $13 + 6$  on one trial and  $6 + 13$  on the next trial. If the children understood that addition was commutative, then they might count "thirteen, fourteen, . . . nineteen" to solve  $13 + 6$ , but then state, without counting, "19" for  $6 + 13$ , and argue "it has the same numbers, it's always the same answer" (p. 160). In this study, 72% of the first graders, and 83% of the second and third graders consistently showed this type of pattern. Using this and a related task, Baroody and Gannon (1984) demonstrated that about 40% of kindergartners recognized commutative relations across formal addition problems. For these kindergarten children and the first graders in the Baroody et al. study, commutative relations were implicitly understood before formal instruction on the topic. Many of the older children explicitly understood that, for example,  $3 + 2 = 2 + 3$ , but it is not clear when and under what conditions (e.g., whether

formal instruction is needed) children come to explicitly understand commutativity as a formal arithmetical principle; that  $a + b = b + a$ .

With respect to associativity, several studies by Canobi and her colleagues (Canobi et al., 1998, 2002) suggest that some kindergarten children recognize associative relations when presented with sets of physical objects, as described, and many first and second graders implicitly understand associative relations when they are presented as addition problems. These studies are also clear in demonstrating that an implicit understanding of associativity does not emerge until after children implicitly understand commutativity, suggesting that implicit knowledge of the latter is the foundation for implicitly understanding the former.

In summary, the conceptual foundation for commutativity emerges from or is reflected in children's manipulation of physical collections of objects and either an inherent or induced understanding that the order in which sets of objects are combined to form a larger collection is irrelevant (R. Gelman & Gallistel, 1978; Resnick, 1992). When presented with sets of physical collections that conform to commutativity, 4- to 5-year-olds recognize the equivalence; that is, they implicitly understand commutativity but they do not explicitly understand commutativity as a formal principle nor do all of the children implicitly understand that commutativity is related to addition at all. Some kindergarteners and most first graders recognize commutative relations in an addition context, but it is not clear when children explicitly understand commutativity as a formal arithmetical principle. Recognition of associative relations with physical sets emerges in some children in kindergarten but only for children who implicitly understand commutativity; an implicit understanding of associativity in an addition context is found for many first and second graders, but it is not known when and under what conditions children come to explicitly understand associativity as a formal arithmetical principle. Even less is known about children's implicit and explicit knowledge of commutativity and associativity as related to multiplication.

**Base-10 Knowledge.** The Hindu-Arabic base-10 system is foundational to modern arithmetic, and thus knowledge of this system is a crucial component of children's developing mathematical competencies. As an example, children's understanding of the conceptual meaning of spoken and written multidigit numerals,

among other features of arithmetic, is dependent on knowledge of the base-10 system (Blöte, Klein, & Beishuizen, 2000; Fuson & Kwon, 1992a). The word *twenty-three* does not simply refer to a collection of 23 objects, it also represents two sets of ten and a set of three unit values. Similarly, the position of the individual numerals (e.g., 1, 2, 3 . . .) in multidigit numerals has a specific mathematical meaning, such that the "2" in "23" represents two sets of 10. Without an appreciation of these features of the base-10 system, children's conceptual understanding of modern arithmetic is compromised.

The base-10 system is almost certainly a biologically secondary competency, the learning of which is not predicted to come easily to children (Geary, 1995, 2002). The more inherent mechanisms that might support learning of the base-10 system are discussed in the "Mechanisms of Change" section. For now, consider that studies conducted in the United States have repeatedly demonstrated that many elementary-school children do not fully understand the base-10 structure of multidigit written numerals (e.g., understanding the positional meaning of the numeral) or number words (Fuson, 1990; Geary, 1994), and thus are unable to effectively use this system when attempting to solve complex arithmetic problems (Fuson & Kwon, 1992a). The situation is similar, though perhaps less extreme, in many European nations (I. Miura et al., 1993). Many of these children have at least an implicit understanding that number-word sequences can be repeated (e.g., counting from 1 to 10 in different contexts) and can of course learn the base-10 system and how to use it in their problem solving. Nonetheless, it appears that many children require instructional techniques that explicitly focus on the specifics of the repeating decade structure of the base-10 system and that clarify often confusing features of the associated notational system (Fuson & Briars, 1990; Varelas & Becker, 1997). An example of the latter is the fact that sometimes 2 represents two units; other times it represents two tens; and still other times it represents two hundreds (Varelas & Becker, 1997).

East Asian students, in contrast, tend to have a better grasp of the base-10 system. I. Miura and her colleagues (I. Miura et al., 1988, 1993) provided evidence supporting their prediction that the earlier described transparent nature of East Asian number words (e.g., "two ten three" instead of "twenty three") facilitates East Asian children's understanding of the structure of the base-10

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system and makes it easier to teach in school (but see Towse & Saxton, 1997). Using a design that enabled the separation of the influence of age and schooling on quantitative and arithmetical competencies, Naito and H. Miura (2001) demonstrated that Japanese children's understanding of the base-10 structure of numerals and their ability to use this knowledge to solve arithmetic problems was largely related to schooling. Because all the children had an advantage of transparent number words, the results suggest teacher-guided instruction on base-10 concepts also facilitates their acquisition (Fuson & Briars, 1990; Varelas & Becker, 1997).

**Fractions.** Fractions are ratios (or part-whole relations) of two or more values and an essential component of arithmetic. They can be represented as proper fractions (e.g.,  $\frac{1}{2}$ ,  $\frac{1}{3}$ ,  $\frac{7}{8}$ ), mixed numbers (e.g.,  $2\frac{1}{3}$ ,  $\frac{3}{4}$ ,  $6\frac{2}{3}$ ), or in decimal form (e.g., 0.5), although they are often represented pictorially during early instruction. As with the base-10 system, research in this area has focused on children's conceptual understanding of fractions, their procedural skills, as in multiplying fractions (Clements & Del Campo, 1990; Hecht, 1998; Hecht, Close, & Santisi, 2003), and the mechanisms that influence the acquisition of these competencies (I. Miura, Okamoto, Vlahovic-Stetic, Kim, & Han, 1999; Rittle-Johnson, Siegler, & Alibali, 2001). When used in a formal mathematical context (e.g., as a decimal), fractions are biologically secondary, although children's experiences with and understanding of part/whole relations among sets of physical objects, such as receiving  $\frac{1}{2}$  of a cookie or having to share one of their two toys, may contribute to a nascent understanding of simple ratios (Mix, Levine, & Huttenlocher, 1999).

In a study of implicit knowledge of simple part-whole relations, Mix et al. (1999) administered a nonverbal task that assessed children's ability to mentally represent and manipulate  $\frac{1}{4}$  segments of a whole circle, and demonstrated that many 4-year-olds recognize fractional manipulations. For example, if  $\frac{3}{4}$  of a circle was placed under a mat and  $\frac{1}{4}$  of the circle was removed, the children recognized that  $\frac{1}{2}$  of a circle remained under the mat. However, it was not until 6 years when children began to understand manipulations that were analogous to mixed numbers; for instance, placing  $1\frac{3}{4}$  circles under the mat and removing  $\frac{1}{2}$  a circle. The results suggest that about the time children begin to show an understanding of part-whole relations in other contexts, as described earlier (Resnick, 1992; Sophian et al., 1995),

they demonstrate a rudimentary understanding of fractional relations. Further, Mix et al., argued these results contradicted predictions based on the numerosity and analog magnitude mechanisms shown in Figure 18.1. This is because R. Gelman (1991) and Wynn (1995) argued these mechanisms only represent whole quantities, and their operation should in fact interfere with children's ability to represent fractions. In this view, fractional representations such as  $\frac{1}{4}$  of a circle can only be understood in terms of discrete numbers, such as 1 and 4, and it is possible that the children in the Mix et al. study represented the  $\frac{1}{4}$  sections of the circles as single units and not as parts of a whole. At this point, the mechanisms underlying children's intuitive understanding of simple fractions, as assessed by Mix et al., remain to be resolved.

Nonetheless, children have considerable difficulty in learning the formal, biologically secondary conceptual and procedural aspects of fractions. During the initial learning of the formal features, such as the meaning of the numerator/denominator notational system, most children treat fractions in terms of their knowledge of counting and arithmetic (Gallistel & Gelman, 1992). A common error when asked to solve, for example,  $\frac{1}{2} + \frac{1}{4}$  is to add the numerators and denominators, yielding an answer of  $\frac{2}{6}$  instead of  $\frac{3}{4}$ . As another example, elementary-school children will often will conclude that since  $3 > 2$ ,  $\frac{1}{3} > \frac{1}{2}$ . The names for fractions may also influence children's early conceptual understanding. In East Asian languages, the part-whole relation of fractions is reflected in the corresponding names; "one-fourth" is "of four parts, one" in Korean. I. Miura et al. (1999) showed these differences in word structure result in East Asian children grasping the part-whole relations represented by simple fractions (e.g.,  $\frac{1}{2}$ ,  $\frac{1}{4}$ ) before formal instruction (in first and second grade) and before children whose native language does not have transparent word names for fractions (Croatian and English in this study). Paik and Mix (2003) demonstrated that first- and second-grade children in the United States perform as well as or better than the same-grade Korean children in the Miura et al. study, when fractions such as  $\frac{1}{4}$  were worded as "one of four parts."

Studies of older elementary-school children and middle-school children have focused on the acquisition of computational skills (e.g., to solve  $\frac{3}{4} \times \frac{1}{2}$ ), conceptual understanding (e.g., that  $2\frac{1}{18} > 1$ ), and the ability to solve word problems involving fractional quantities (Byrnes & Wasik, 1991; Rittle-Johnson

et al., 2001). Hecht (1998) assessed the relation between children's accuracy at recognizing formal procedural rules for fractions (e.g., when multiplying, the numerators are multiplied and the denominators are multiplied), their conceptual understanding of fractions (e.g., of part-whole relations), their basic arithmetic skills and their ability to solve computational fraction problems, word problems that include fractions, and to estimate fractional quantities. Among other findings, the results demonstrated that scores on the procedure-recognition test predicted computational skills (i.e., accuracy in adding, multiplying, and dividing proper and mixed fractions), above and beyond the influence of IQ, reading skills, and conceptual knowledge. And, conceptual knowledge predicted accuracy at solving word problems that involved fractions and especially accuracy in the estimation task, above and beyond these other influences.

In a follow-up study, Hecht et al. (2003) demonstrated the acquisition of conceptual knowledge of fractions and basic arithmetic skills were related to children's working memory capacity and their on-task time in math class. Conceptual knowledge and basic arithmetic skills, in turn, predicted accuracy at solving computational problems involving fractions; conceptual knowledge and working memory predicted accuracy at setting up word problems that included fractions; and conceptual knowledge predicted fraction estimation skills. Rittle-Johnson et al. (2001) demonstrated that children's skill at solving decimal fractions was related to both their procedural and their conceptual knowledge of fractions and that learning procedural and conceptual knowledge occurred iteratively. Good procedural skills predicted gains in conceptual knowledge and vice versa (Sophian, 1997). The mechanism linking procedural and conceptual knowledge appeared to be the children's ability to represent the decimal fraction on a mental number line.

In all, these studies suggest that preschool and early elementary-school children have a rudimentary understanding of simple fractional relationships, but the mechanism underlying this knowledge is not yet known. It might be the case that the ability to visualize simple part-whole relations (e.g.,  $\frac{1}{2}$  of natural objects that can be decomposed, as in eating an apple) is biologically primary. Difficulties arise when children are introduced to biologically secondary features of fractions, specifically, as formal mathematical principles and as formal procedures that can be used in problem-solving contexts.

A source of difficulty is the misapplication of counting knowledge and arithmetic with whole numbers to fractions. Children's conceptual understanding of fractions and their knowledge of associated procedures emerges slowly, sometimes not at all, during the elementary and middle school years. The mechanisms that contribute to the emergence of these biologically secondary competencies are not fully understood, but involve instruction, working memory, and the bidirectional influences of procedural knowledge on the acquisition of conceptual knowledge and conceptual knowledge on the skilled use of procedures.

**Estimation.** Although not a formal content domain, the ability to estimate enables individuals to assess the reasonableness of solutions to mathematical problems and thus it is an important competency within many mathematical domains. Estimation typically involves some type of procedure to generate an approximate answer to a problem when calculation of an exact answer is too difficult or is unnecessary. The study of children's ability to estimate in school-taught domains and the development of these skills has focused on computational arithmetic (Case & Okamoto, 1996; Dowker, 1997, 2003; LeFevre, Greenham, & Waheed, 1993; Lemaire & Lecacheur, 2002b) and the number line (Siegler & Booth, 2004, 2005; Siegler & Opfer, 2003). Studies in both of these areas indicate that the ability to generate reasonable estimates is difficult for children and some adults and only appears with formal schooling.

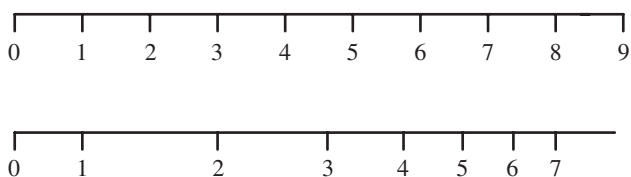
Dowker (1997, 2003) demonstrated that elementary-school children's ability to generate reasonable estimates to arithmetic problems (e.g.,  $34 + 45$ ) is linked to their computational skills. More specifically, children can estimate within a zone of partial knowledge that extends just beyond their ability to mentally calculate exact answers. Children who can solve addition problems with sums less than 11 (e.g.,  $3 + 5$ ) can estimate answers to problems with sums less than about 20 (e.g.,  $9 + 7$ ), but guess for larger-valued problems (e.g.,  $13 + 24$ ). Children who can solve multidigit problems without carries (e.g.,  $13 + 24$ ) can estimate answers to problems of the same magnitude with carries (e.g.,  $16 + 29$ ), but guess for larger-valued problems (e.g.,  $598 + 634$ ). Studies of how children and adults estimate have revealed that, as with most cognitive domains (Siegler, 1996), they use a variety of strategies (LeFevre et al., 1993; Lemaire & Lecacheur, 2002b). For younger children, a common strategy for addition is to round the units or

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decades segments down to the nearest decade or hundreds value and then add, as in  $30 + 50 = 80$  to estimate  $32 + 53$ , or  $200 + 600 = 800$  to estimate  $213 + 632$ . Older children and adults use more sophisticated strategies than younger children, and may for instance compensate for the initial rounding to adjust their first estimate. An example would involve adding  $10 + 30 = 40$  to the initial estimate of 800 for the  $213 + 632$  problem.

Siegler and his colleagues (Siegler & Booth, 2004; Siegler & Opfer, 2003) have studied how children estimate magnitude on a number line, as in estimating the position of 83 on a 1 to 100 number line. As with arithmetic, young children are not skilled at making these estimates but improve considerably during the elementary school years, and come to use a variety of strategies for making reasonable estimates. The most intriguing finding is that young children's performance suggests their estimates are supported by the analog magnitude mechanism described in the "Early Quantitative Abilities" section. Figure 18.2 provides an illustration of how this mechanism would map onto the number line. The top number line illustrates the values across the first 9 integers as defined by the formal mathematical system. The distance between successive numbers and the quantity represented by this distance are precisely the same at any position in the number line (e.g., 2 3, versus 6 7, or versus 123 124). The bottom number line illustrates perceived distances predicted to result from estimates based on the analog magnitude mechanism (Dehaene, 1997; Gallistel & Gelman, 1992). As magnitude increases, the number line is "compressed" such that distances between smaller-valued integers, such as 2 to 3, are more salient and more easily discriminated than distances associated with larger-valued integers.

When kindergarten children are asked to place numbers on a 1 and 100 number line, they place 10 too far to the right and compress the placement of larger-valued



**Figure 18.2** Spatial representations of number-line magnitudes. The top is the standard mathematical number line, and the bottom is the number line generated when numbers are mapped onto the analog magnitude mechanism.

numbers (Siegler & Booth, 2004), in keeping with the view that their initial understanding of Arabic numerals maps onto the analog magnitude mechanism (Spelke, 2000). By the end of elementary school, most children develop the correct mathematical representation of number placement and the meaning of this placement (Siegler & Opfer, 2003). Dehaene (1997), Gallistel and Gelman (1992), and Siegler and Booth (2005) proposed the analog magnitude mechanism onto which the mathematical number line is mapped is spatial. Indeed, Zorzi, Priftis, and Umiltá (2002) found that individuals with injury to the right-parietal cortex showed deficits in spatial orientation and number line estimation. Dehaene et al. (1999) showed adults' computational estimation may also be dependent on a similar spatial system, that is, a mental number line.

### *Arithmetic Operations*

Research in this area is focused on children's explicit and goal-directed solving of arithmetic problems, as contrasted with an implicit understanding of the effects of addition and subtraction on the quantity of small sets. In the first section, I provide an overview of arithmetical development across cultures and in the second focus on addition to more fully illustrate developmental and schooling-based changes in how children solve arithmetic problems. In the third section, I provide a few examples of how children solve subtraction, multiplication, and division problems (for a more thorough discussion, see Geary, 1994); and in the final section I discuss the relation between conceptual knowledge and procedural skills in arithmetic.

**Arithmetic across Cultures.** Even before formal instruction, there are important cross-cultural similarities in children's arithmetical development. Within all cultural groups that have been studied and across all four arithmetic operations, development is not simply a matter of switching from a less mature problem-solving strategy to a more adult-like strategy. Rather, at any given time most children use a variety of strategies to solve arithmetic problems (Siegler, 1996). They might count on their fingers to solve one problem, retrieve the answer to the next problem, and count verbally to solve still other problems. Arithmetic development involves a change in the mix of strategies, as well as changes in the accuracy and speed with which each strategy can be executed. A more intriguing finding is that children do not

randomly choose one strategy, such as counting to solve one problem, and then retrieval to solve the next problem. Rather, children “often choose among strategies in ways that result in each strategy’s being used most often on problems where the strategy’s speed and accuracy are advantageous, relative to those of other available procedures” (Siegler, 1989, p. 497).

With instruction, a similar pattern of developmental change in strategy usage emerges for children in different cultures, although the rate with which the mix of strategies changes varies from one culture to the next (Geary et al., 1996; Ginsburg, Posner, & Russell, 1981; Saxe, 1985; Svenson & Sjöberg, 1983). Children in China shift from one favored strategy to the next at a younger age than do children in the United States, whereas the Oksapmin appear to shift at later ages (Geary et al., 1996; Saxe, 1982). These cultural differences appear to reflect differences in the amount of experience with solving arithmetic problems (Stevenson, Lee, Chen, Stigler, Hsu, & Kitamura, 1990), and to a lesser extent the earlier described language differences in number words. As an example, Ilg and Ames (1951) provided a normative study of arithmetic development for American children between the ages of 5 and 9 years. These children received their elementary-school education at a time when basic skills were emphasized much more than today. The types of skills and problem-solving strategies they described for American children at that time was very similar to arithmetic skills we see in present day same-age East Asian children (Fuson & Kwon, 1992b; Geary et al., 1996). Today, however, the arithmetic skills of East Asian children are much better developed than those of their same-age American peers (see also Geary et al., 1997).

**Addition Strategies.** The most thoroughly studied developmental and schooling-based improvement in arithmetical competency is change in the distribution of procedures, or strategies, children use during problem solving (e.g., Ashcraft, 1982; Carpenter & Moser, 1984; Geary, 1994; Lemaire & Siegler, 1995; Siegler & Shrager, 1984). Some 3-year-olds can explicitly solve simple addition problems and do so based on their counting skills (Fuson, 1982; Saxe et al., 1987). A common approach is to use objects or manipulatives. If the child is asked “how much is one cookie and two cookies,” she will typically count out two objects, then one object, and finally all of the objects are counted starting from one.

The child states, while pointing at each object in succession, “one, two, three.” The use of manipulatives serves at least two purposes. First, the sets of objects represent the numbers to be counted. The meaning of the abstract numeral, “three” in this example, is literally represented by the objects. Second, pointing to the objects during counting helps the child to keep track of the counting process (Carpenter & Moser, 1983; R. Gelman & Gallistel, 1978). The use of manipulatives is even seen in some 4- and 5-year-olds, depending on the complexity of the problem (Fuson, 1982).

By the time children enter kindergarten, a common strategy for solving simple addition problems is to count both addends. These counting procedures are sometimes executed with the aid of fingers, the finger-counting strategy, and sometimes without them, the verbal counting strategy (Siegler & Shrager, 1984). The two most commonly used counting procedures, whether children use their fingers or not, are called *min* (or counting on) and *sum* (or counting all; Fuson, 1982; Groen & Parkman, 1972). The *min* procedure involves stating the larger-valued addend and then counting a number of times equal to the value of the smaller addend, such as counting 5, 6, 7, 8 to solve  $5 + 3$ . With the *max* procedure, children start with the smaller addend and count the larger addend. The *sum* procedure involves counting both addends starting from 1. The development of procedural competencies is related, in part, to improvements in children’s conceptual understanding of counting and is reflected in a gradual shift from frequent use of the *sum* procedure to the *min* procedure (Geary, Bow-Thomas, & Yao, 1992; Siegler, 1987).

The use of counting also results in the development of memory representations of basic facts (Siegler & Shrager, 1984). Once formed, these long-term memory representations support the use of memory-based problem-solving processes. The most common of these are direct retrieval of arithmetic facts and decomposition. With direct retrieval, children state an answer that is associated in long-term memory with the presented problem, such as stating “eight” when asked to solve  $5 + 3$ . Decomposition involves reconstructing the answer based on the retrieval of a partial sum. The problem  $6 + 7$  might be solved by retrieving the answer to  $6 + 6$  and then adding 1 to this partial sum. With the fingers strategy, children raise a number of fingers to represent the addends, which appears to trigger retrieval of the answer. The use of retrieval-based processes is moderated

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by a confidence criterion that represents an internal standard against which the child gauges confidence in the correctness of the retrieved answer. Children with a rigorous criterion only state answers that they are certain are correct, whereas children with a lenient criterion state any retrieved answer, correct or not (Siegler, 1988a). The transition to memory-based processes results in the quick solution of individual problems and reductions in the working memory demands that appear to accompany the use of counting procedures (Delaney, Reder, Staszewski, & Ritter, 1998; Geary et al., 1996; Lemaire & Siegler, 1995).

The general pattern of change is from use of the least sophisticated problem-solving procedures, such as sum counting, to the most efficient retrieval-based processes. The relatively unsophisticated counting-fingers strategy, especially with use of the sum procedure, is heavily dependent on working memory resources and takes the most time to execute, whereas direct retrieval requires little of working memory and is executed quickly, and with practice accurately and automatically (Geary, Hoard, Byrd-Craven, & DeSoto, 2004). However, as noted, development is not simply a switch from use of a less sophisticated strategy to use of a more sophisticated strategy. Rather, the change is best captured by Siegler's (1996) overlapping waves metaphor of cognitive development, as noted earlier. An example is provided by Geary et al.'s (1996) assessment of the strategies used by first-, second-, and third-grade children in China and the United States to solve simple addition problems. In the fall of first grade, the Chinese and American children, as groups and as individuals, used a mix of counting and retrieval to solve these problems. The American children used finger counting and verbal counting, but the Chinese children rarely used finger counting, although they did use this strategy in kindergarten. Across grades, there is an increase in use of direct retrieval and a decrease in counting for both groups, but a much more rapid (within and across school years) change in the strategy mix for the Chinese children.

When solving more complex addition problems, children initially rely on the knowledge and skills acquired for solving simple addition problems (Siegler, 1983). Strategies for solving more complex problems include counting, decomposition or regrouping, as well as the formally taught columnar procedure (Fuson, Stigler, & Bartsch, 1988; Ginsburg, 1977). Counting strategies typically involve counting on from the larger number (Siegler & Jenkins, 1989). For instance,  $23 + 4$  would be

solved by counting "twenty-three, twenty-four, twenty-five, twenty-six, twenty-seven." The decomposition or regrouping strategy involves adding the tens values and the units values separately. So the problem  $23 + 45$  would involve the steps  $20 + 40$ ,  $3 + 5$ , and then  $60 + 8$  (Fuson & Kwon, 1992a). The most difficult process involves carrying or trading, as in the problem  $46 + 58$ . These problems are difficult because of the working memory demands of trading (Hamann & Ashcraft, 1985; Hitch, 1978; Widaman, Geary, Cormier, & Little, 1989), and because trading requires an understanding of place value and the base-10 system.

**Subtraction, Multiplication, and Division Strategies.** Many of the same strategic and development trends described for children's addition apply to children's subtraction (Carpenter & Moser, 1983, 1984). Early on, children use a mix of strategies but largely count, often using manipulatives and fingers to help them represent the problem and keep track of the counting (Saxe, 1985). With experience and improvements in working memory, children are better able to mentally keep track of the counting process, and thus gradually abandon the use of manipulatives and fingers for verbal counting. There are two common counting procedures, counting up or counting down. Counting up involves stating the value of the subtrahend (bottom number), and then counting until the value of the minuend (top number) is reached. For example,  $9 - 7$  is solved by counting "eight, nine." Since two numbers were counted, the answer is two. Counting down is often used to solve more complex subtraction problems, such as  $23 - 4$  (Siegler, 1989), and involves counting backward from the minuend a number of times represented by the value of the subtrahend. Children also rely on their knowledge of addition facts to solve subtraction problems, which is called addition reference ( $9 - 7 = 2$ , because  $7 + 2 = 9$ ). The most sophisticated procedures involve decomposing the problems into a series of simpler problems (Fuson & Kwon, 1992a), or solving the problem using the school-taught columnar approach.

Developmental trends in children's simple multiplication mirror the trends described for children's addition and subtraction, although formal skill acquisition begins in the second or third grade, at least in the United States (Geary, 1994). The initial mix of multiplication strategies is grounded in children's knowledge of addition and counting (Campbell & Graham, 1985; Siegler, 1988b). These strategies include repeated addition and

counting by  $n$ . Repeated addition involves representing the multiplicand a number of times represented by the multiplier, and then successively adding these values; for example, adding  $2 + 2 + 2$  to solve  $2 \times 3$ . The counting by  $n$  strategy is based on the child's ability to count by 2s, 3s, 5s, and so on. Somewhat more sophisticated strategies involve rules, such as  $n \times 0 = 0$ , and decomposition (e.g.,  $12 \times 2 = 10 \times 2 + 2 \times 2$ ). These procedures result in the formation of problem/answer associations such that most children are able to retrieve most multiplication facts from long-term memory by the end of the elementary school years (Miller & Paredes, 1990). Considerably less research has been conducted on children's strategies for solving complex multiplication problems, but the studies that have been conducted with adults suggest that many individuals eventually adopt the formally taught columnar procedure (Geary, Widaman, & Little, 1986).

The first of two classes of strategy used for solving division problems are based on the child's knowledge of multiplication (Ilg & Ames, 1951; Vergnaud, 1983). For example, the solving of  $20/4$  (20 is the dividend, and 4 is the divisor) is based on the child's knowledge that  $5 \times 4 = 20$ , which is called multiplication reference (see also Campbell, 1999). For children who have not yet mastered the multiplication table, a derivative of this strategy is sometimes used. Here, the child multiplies the divisor by a succession of numbers until she finds the combination that equals the dividend. To solve  $20/4$ , the sequence for this strategy might be  $4 \times 2 = 8$ ,  $4 \times 3 = 12$ ,  $4 \times 4 = 16$ ,  $4 \times 5 = 20$ . The second class of strategies is based on the child's knowledge of addition. The first involves a form of repeated addition. To solve  $20/4$ , the child would produce the sequence  $4 + 4 + 4 + 4 + 4 = 20$ , and then count the fours. The number of counted fours represents the quotient. Sometimes children try to solve division problems based directly on their knowledge of addition facts. For instance, to solve the problem  $12/2$ , the child might base her answer on the knowledge that  $6 + 6 = 12$ ; thus 12 is composed of two 6s.

**Conceptual Knowledge.** The progression in the mix of problem-solving strategies described in the preceding sections is related to frequency of exposure to the problems, instruction, and to children's understanding of related concepts (Blöte et al., 2000; Geary et al., 1996; Klein & Bisanz, 2000; Siegler & Stern, 1998). For example, children's use of min counting, as contrasted with sum counting, as well as their use of decomposition

to solve simple and complex addition problems is related to their understanding of commutativity (Canobi et al., 1998; R. Cowan & Renton, 1996) and to their understanding of counting principles (Geary et al., 2004). Children's understanding of the base-10 system facilitates their understanding of place value and is related to the ease with which they learn to trade from one column to the next when solving complex addition and subtraction problems (e.g.,  $234 + 589$ ;  $82 - 49$ ) and to the frequency with which they commit trading errors (Fuson & Kwon, 1992a). For trading in the problem  $34 + 29$ , children must understand that the 1 traded from the units- to the tens-column represents 10 and not 1. Failure to understand this contributes to common trading errors, as in  $34 + 29 = 513$  where  $4 + 9$  is written as 13 without the trade (Fuson & Briars, 1990).

### *Arithmetical Problem Solving*

The study of children's and adolescents' mathematical problem solving typically focuses on competence in solving arithmetical or algebraic word problems, and the cognitive processes (Hegarty & Kozhevnikov, 1999; Mayer, 1985) and instructional factors (Fuchs et al., 2003a; Sweller, Mawer, & Ward, 1983) that drive the development of this competence. The ability to solve all but the most simple arithmetical and algebraic word problems does not come easily to children or many adults, and represents an important biologically secondary mathematical competence. A full review is beyond the scope of this chapter (see Geary, 1994; Mayer, 1985; Tronsky & Royer, 2002), but I illustrate some of the core issues as related to the solving of arithmetical word problems.

**Problem Schema.** Word problems can be placed in categories based on the relations among objects, persons, or events described in the problem and the types of procedures needed to solve the problem (G. Cooper & Sweller, 1987; Mayer, 1985). The relations and procedures that define each category compose the schema for the problem type. As shown in Table 18.1, most simple word problems that involve addition or subtraction can be classified into four general categories: change, combine, compare, and equalize (Carpenter & Moser, 1983; De Corte & Verschaffel, 1987; Riley et al., 1983). Change problems imply some type of action be performed by the child, and are solved using the same procedures children use to solve standard arithmetic problems (e.g.,  $3 + 5$ ; Jordan & Montani, 1997). Most

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kindergarten and first-grade children can easily solve the first type of change problem shown in the table. Children typically represent the meaning of the first term, “Andy had two marbles,” through the use of two blocks or uplifted fingers. The meaning of the next term, “Nick gave him three more marbles,” is represented by three blocks or uplifted fingers. Some children then answer the “how many” question by literally moving the two sets of blocks together and then counting them or counting the total number of uplifted fingers (Riley & Greeno, 1988).

The change and combine problems are conceptually different, even though the basic arithmetic is the same

(Briars & Larkin, 1984; Carpenter & Moser, 1983; Riley et al., 1983). Combine problems involve a static relationship, rather than the implied action found with change problems. The first examples under the change and combine categories presented in Table 18.1 illustrate the point. Both problems require the child to add  $2 + 3$ . In the change problem, the quantities in Andy’s and Nick’s sets differ after the action (addition) has been performed. In the combine example, however, the status—the quantity of owned marbles—of Andy’s set and Nick’s set is not altered by the addition. Compare problems involve the same types of static relationships found in combine problems. The quantity of the sets does not change. Rather, the arithmetic operation results in determining the exact quantity of one of the sets by reference to the other set, as illustrated with the third and fourth compare examples in Table 18.1. The first and second compare examples involve a more straightforward greater than/less than comparison. Examination of the equalize problems presented in Table 18.1 reveals that they are conceptually similar to change problems. The action, or arithmetic, performed results in a change in the quantity of one of the sets. The change is constrained such that the result is equal sets once the action has been completed, whereas there is no such constraint with change problems.

Across categories, the complexity of word problems increases with increases in the number of steps or procedures needed to solve the problem; the complexity of these procedures (e.g., differentiation in calculus versus simple addition); and the complexity of the relations among persons, objects, or events (e.g., timing of two events) conveyed in the problem. These more complex problems can still be categorized in terms of schemas, that is, the specific nature of the relations among these persons, objects, or events and the types of procedures needed to solve the problem (Sweller et al., 1983; Mayer, 1981).

**Problem-Solving Processes.** The processes involved in solving word problems include translating each sentence into numerical or mathematical information, integrating the relations among this information, and executing the necessary procedure or procedures (Briars & Larkin, 1984; Kintsch & Greeno, 1985; Mayer, 1985; Riley et al., 1983; Riley & Greeno, 1988). For complex, multistep problems and especially unfamiliar problems, an additional step is solution planning—making explicit metacognitive decisions about problem type and potential procedural steps for solving the problem

**TABLE 18.1** Classification of Arithmetic Word Problems

Change
1. Andy had two marbles. Nick gave him three more marbles. How many marbles does Andy have now?
2. Andy had five marbles. Then he gave three marbles to Nick. How many marbles does Andy have now?
3. Andy had two marbles. Nick gave him some more marbles. Now Andy has five marbles. How many marbles did Nick give him?
4. Nick had some marbles. Then he gave two marbles to Andy. Now Nick has three marbles. How many marbles did Nick have in the beginning?
Combine
1. Andy has two marbles. Nick has three marbles. How many marbles do they have altogether?
2. Andy has five marbles. Three are red marbles and the rest are blue marbles. How many blue marbles does Andy have?
Compare
1. Nick has three marbles. Andy has two marbles. How many fewer marbles does Andy have than Nick?
2. Nick has five marbles. Andy has two marbles. How many more marbles does Nick have than Andy?
3. Andy has two marbles. Nick has one more marble than Andy. How many marbles does Nick have?
4. Andy has two marbles. He has one marble less than Nick. How many marbles does Nick have?
Equalize
1. Nick has five marbles. Andy has two marbles. How many marbles does Andy have to buy to have as many marbles as Nick?
2. Nick has five marbles. Andy has two marbles. How many marbles does Nick have to give away to have as many marbles as Andy?
3. Nick has five marbles. If he gives away three marbles, then he will have as many marbles as Andy. How many marbles does Andy have?
4. Andy has two marbles. If he buys one more marble, then he will have the same number of marbles as Nick. How many marbles does Nick have?
5. Andy has two marbles. If Nick gives away one of his marbles, then he will have the same number of marbles as Andy. How many marbles does Nick have?

(Fuchs, Fuchs, Prentice, Burch, Hamlett, Owen, & Schroeter, 2003; Mayer, 1985; Schoenfeld, 1987). The latter is important, because the number of potential ways a problem can be solved (e.g., sequence in which procedures are executed) increases with increases in problem complexity. The translation process is straightforward for most of the problems shown in Table 18.1, but integrating this information across sentences can be more difficult. Consider the second compare problem. Here, the key word “more” will often prompt children to add the quantities. These types of error can often be avoided if, for example, the relations between the quantities held by Nick and Andy are concretely represented on a visuospatial number line. Once represented in this way, the child can easily count from 2 to 5 on the number line and correctly determine that Nick has three more than Andy.

Improvement in the ability to execute these problem-solving processes and thus the ability to solve complex word problems is related to intraindividual and extraindividual factors. The latter include the quantity (e.g., amount of practice) and form of instruction (Fuchs, Fuchs, Prentice, Burch, Hamlett, Owen, & Schroeter, 2003; Sweller et al., 1983), and more general curricular factors, such as the grade in which multistep problems are introduced and the quality of textbooks (Fuson et al., 1988). One of the more difficult competencies to teach is the ability to transfer knowledge from familiar to novel problems. Teaching of this competence often requires explicit instruction on how to make links (e.g., based on problem type) between familiar and novel problems (Fuchs et al., 2003). For children and adults, intraindividual factors include working memory capacity (Geary & Widaman, 1992; Tronsky & Royer, 2002), basic computational skills (Sweller et al., 1983), knowledge of problem type and schema (G. Cooper & Sweller, 1987), the ability to form visuospatial representations of relations among problem features (Hegarty & Kozhevnikov, 1999; Lewis & Mayer, 1987), as well as self-efficacy and the resultant tendency to persist with problem-solving efforts (Fuchs, Fuchs, Prentice, Burch, Hamlett, Owen, & Schroeter, 2003).

### **Mechanisms of Change**

The processes that result in variability, competition, and selection drive change at all levels, ranging from gene frequency in populations to the pattern of corporate successes and failures. The combination of variability,

competition, and selection is thus a useful organizing framework for studying the mechanisms underlying the development of children’s mathematical competence and for understanding cognitive development in general (Siegler, 1996). Developmental or experience-related improvements in cognitive competence in turn are indexed by efficiency in achieving goals, such as solving an arithmetic problem or generating plans for the weekend, with efficiency typically measured in terms of the speed and accuracy with which the goal or subgoals are achieved. From this perspective, developmental or experience-related improvement in goal achievement will be related, in part, to (a) variability in the procedures or concepts that can be employed to achieve the goal; (b) underlying brain and cognitive mechanisms that generate this variability; and (c) contextual as well as brain and cognitive feedback mechanisms that select the most efficient procedures or concepts.

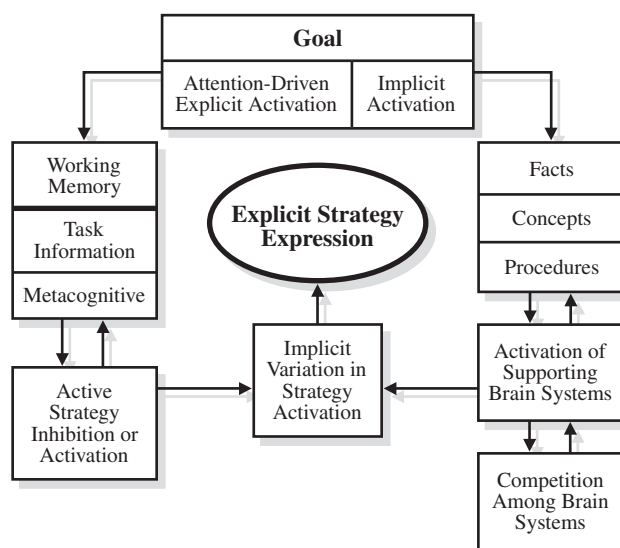
In the first section, I outline some basic mechanisms related to variability, competition, and selection among different problem-solving approaches; more detailed discussion of associated brain (Edelman, 1987) and cognitive (Shrager & Siegler, 1998; Siegler, 1996) mechanisms can be found elsewhere. In the second section, I conjecture about developmental change in these mechanisms, as related to biologically primary and biologically secondary competencies (Geary, 2004a, 2005). The discussion in both sections is necessarily preliminary but should provide an outline for conceptualizing how change might occur in mathematical and other domains.

### ***Darwinian Competition***

**Variability and Competition.** Siegler and his colleagues have demonstrated that children use a variety of procedures and concepts to solve problems in nearly all cognitive domains, and that with development and experience they become more efficient in achieving problem-solving goals (Shrager & Siegler, 1998; Siegler, 1996; Siegler & Shipley, 1995). The issue at this point is not whether variability in problem-solving approaches is an integral feature of cognitive development, but rather the mechanisms that result in variability and competition among these approaches. In Figure 18.3, I provide a skeletal framework for conceptualizing the mechanisms that might contribute to variability in, and competition among, problem-solving approaches at the level of cognitive and brain systems.

First, goal achievement requires an attentional focus on features of the external context that are either related

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**Figure 18.3** Outline of explicit and implicit mechanisms that appear to contribute to the explicit use of one problem-solving strategy or another (see text for detailed description).

to the goal (e.g., a sheet of arithmetic problems to solve) or to an internally generated mental model of an anticipated goal-related context (Geary, 2005). Goal generation and a narrowing of attention onto the features of the external or internally represented context will result in the implicit activation—above baseline levels of activation but outside conscious awareness—of goal-related facts, concepts, and procedures (Anderson, 1982). These can be learned facts, concepts, and procedures associated with previous problem solving (Shrager & Siegler, 1998; Siegler, 1996) or inherent but rudimentary concepts and procedures (R. Gelman, 1990, 1991). The implicit activation of goal-related information means that existing conceptual knowledge and memorized facts and procedures may influence where attention is allocated and thus influence problem solving. Knowledge of the base-10 system and memorized procedures for solving complex arithmetic problems should influence the temporal pattern of attentional focus, from the units-column numbers to the tens-column numbers, and so on. More generally, conceptual knowledge can be understood at multiple levels, ranging from explicit descriptors of categorical features (e.g., sets of ten) to patterns of brain activity that direct attention toward specific goal-related features of the environment (e.g., columnar numbers, eyes on a human face; Schyns, Bonnar, & Gosselin, 2002).

The implicit activation of goal-related information will often mean that multiple facts, concepts, or proce-

dures are simultaneously activated and, as indicated by the bidirectional arrows in the right-hand side of Figure 18.3, multiple underlying systems of brain regions will be simultaneously activated. Edelman (1987) proposed these activated populations of neurons compete, as integrated groups, for downstream explicit, behavioral expression. As is also shown in the right-hand side of Figure 18.3, competition at the level of brain systems—relative activation of neuronal groups and inhibition of competing groups—creates variation in the level of implicit activation of corresponding facts, concepts, and procedures associated with goal-related behavioral strategies and thus variation in the probability of the expression of one strategy or another. Edelman also proposed that normal neurodevelopmental processes create a pattern of brain organization that will necessarily result in variation and competition at the level of neuronal groups and thus variation and competition at the levels of perception, cognition, and behavioral expression. The resulting systems of neuronal groups are well understood for only a few restricted perceptual or cognitive phenomena related to classical and operant conditioning (e.g., McDonald & White, 1993; Thompson & Krupa, 1994), although they are beginning to be explored in mathematical domains (Dehaene et al., 1999; Pinel et al., 2004). The point is that competition among neuronal groups activated by goal-related contextual features creates variation at the level of activation of cognitive representations (e.g., long-term memories of goal-related facts) and thus variation in the probability that one strategy or another will be expressed.

It is also clear that attentional focus can result in an explicit and conscious representation of the goal and goal-related facts, concepts, and procedures in working memory (N. Cowan, 1995; Dehaene & Naccache, 2001; Engle, 2002; Shrager & Siegler, 1998), as shown in the left-hand side of Figure 18.3. The working memory system provides at least two strategy-selection mechanisms. The first is composed of explicitly represented task or metacognitive knowledge that can guide the strategy execution and enable inferences to be drawn about goal-related concepts or procedures based on observation of problem-solving processes and outcomes. A task-relevant inference was provided by Briars' and Siegler's (1984) finding that preschool children's knowledge of counting is derived, in part, through observation of culture-specific procedures, such as counting from left to right. Metacognitive mechanisms operate across tasks and represent the ability to monitor problem solving for ways to improve efficiency. An example is chil-

dren's discovery of min counting (Siegler & Jenkins, 1989). Although many children switch from sum counting to min counting before they are explicitly aware they have done so, others correctly infer that you do not have to count both addends to solve the problem and adjust their problem solving accordingly. In other words, explicit monitoring of goal-related problem solving can result in the identification and elimination of unnecessary problem-solving steps.

The second way in which explicit attentional control can influence problem solving is through active strategy inhibition or activation. An example is provided by choice/no choice problem-solving instructions (Siegler & Lemaire, 1997). Here, participants are instructed to solve one set of problems (e.g., arithmetic problems, spelling items) using any strategy they choose (e.g., finger counting, or retrieval), and are instructed to solve a comparable set of problems using only a single strategy, such as direct retrieval. Problem solving is more accurate under choice conditions, in keeping with the prediction that strategy variability is adaptive (Siegler, 1996). Equally important, the ability to explicitly inhibit the expression of all but one problem-solving strategy is easily achieved by most elementary-school children (Geary, Hamson, & Hoard, 2000; Jordan & Montani, 1997; Lemaire & Lecacheur, 2002a). Brain-imaging studies suggest that the dorsolateral prefrontal cortex is involved in attentional control, monitoring of explicit problem-solving, and the inhibition of task-irrelevant brain regions (Kane & Engle, 2002); maturation of these brain regions may contribute to developmental change in the ability to consciously influence strategy choices (Welsh & Pennington, 1988). As shown in the bottom left-hand side of Figure 18.3, these brain regions can amplify or inhibit the implicitly activated brain regions represented at the right-hand side of the figure. The combination of bottom-up competition and top-down mechanisms that amplify or inhibit these competing systems results in the expression of one behavioral strategy or another.

**Selection.** Variability and competition at levels of brain and cognition result in the execution of one problem-solving strategy or another. Over time, adaptive goal-related adjustments lead to the most efficient strategies being used more often and less efficient strategies becoming extinct (see Siegler, 1996 for review). Given this pattern, the *consequence* (e.g., successful problem solving) and *process* of strategy execution must be among the selection mechanisms acting on indi-

vidual strategies. The specific means through which these selection processes operate are not well understood, but must involve changes in the pattern of memory and concept representation at both the brain (Edelman, 1987) and cognitive (Siegler, 1996) levels. Just as natural selection results in changes in gene frequencies across generations, selection processes that act on strategic variability result in a form of inheritance: changes in memory patterns, strategy knowledge, and physical changes in the underlying neuronal groups.

At a behavioral level, the most fundamental selective mechanisms are classical and operant conditioning (Timberlake, 1994), as illustrated by the learning of simple addition (Siegler & Robinson, 1982). As noted, early in skill development, children rely on finger and verbal counting. The repeated execution of these procedures has at least two effects. First, the procedures themselves are executed more quickly and with greater accuracy (Delaney et al., 1998), indicating the formation and strengthening of procedural memories. Second, the act of counting out the addends creates an associative link between the addend pair and the generated answer, that is, the formation of a declarative memory (Siegler & Shrager, 1984; see also Campbell, 1995). The neural changes that result in the formation and strengthening of these procedural and declarative memories are not known, but may require a simultaneous and synchronized firing of neurons within the associated neuronal group (e.g., Damasio, 1989). At a cognitive level, the formation of the associative memory results in greater variation in the number of processes (e.g., counting and retrieval) that can be used to achieve the goal and thus increases competition among these processes for expression.

Empirically, it is known that direct retrieval of arithmetic facts eventually obtains a selective advantage over the execution of counting procedures. Speed of retrieval (Siegler & Shrager, 1984) and the lower working memory demands of direct retrieval versus procedural execution (Geary et al., 2004) almost certainly contribute to this selective advantage. With sufficient experience, the speed of retrieval increases and the generated answer tends to be as accurate or more accurate than that generated by counting procedures. In effect, the greater efficiency of retrieval means that the goal is achieved before the execution of alternative processes can be completed. Goal achievement in turn may act—in ways not yet understood—to inhibit the expression of these alternative processes and may further enhance the competitive advantage of direct retrieval. At the level of

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brain systems, Edelman (1987) predicted that these selection mechanisms will strengthen connections among neurons within the selected neuronal group supporting retrieval in this example, and result in the death of neurons or at least fewer connections among the neurons within the neuronal groups supporting strategies that are not expressed (counting in this example).

### *Primary and Secondary Domains*

**Primary.** Primary competencies are composed of an inherent, though often not fully developed, system of implicit concepts, procedures, and supporting brain systems represented at the right-hand side of Figure 18.3. The result is attentional, affective, and information-processing biases that orient the child toward the features of the ecology that were of significance during human evolution (Geary & Bjorklund, 2000). Coupled with these biases are self-initiated behavioral engagements of the ecology (Bjorklund & Pellegrini, 2002; Scarr, 1992). The latter generate evolutionarily expectant experiences that provide the feedback needed to adjust the architecture of primary brain and cognitive systems to nuances in evolutionarily significant domains (Greenough, Black, & Wallace, 1987), such as allowing the individual to discriminate one face from another. These behavioral biases are expressed as common childhood activities, such as social play and exploration of the environment.

Proposed biologically primary mathematical competencies include an implicit understanding of the quantity of small sets (Starkey et al., 1990); the effects of addition and subtraction on quantity (Wynn, 1992a); counting concepts and how to execute counting procedures (R. Gelman & Gallistel, 1978); and relative quantity (Lipton & Spelke, 2003). There is little developmental change in some of these primary competencies, such as the ability to estimate the quantity of small sets (Geary, 1994; Temple & Posner, 1998), and substantive developmental improvement in other competencies, such as skill at executing counting procedures and the emergence of an explicit understanding of some counting concepts (R. Gelman & Gallistel, 1978). Developmental change in these latter competencies is predicted to emerge from an interaction between early attentional biases and related activities such as counting sets of objects (R. Gelman, 1990). These activities allow inherent biases to be integrated with culturally specific practices, such as the culture's number words or manner of representing counts (e.g., on fin-

gers or other body parts, as with the Oksapmin). By the end of the preschool years, all these competencies appear to be well integrated and allow children to perform basic measurements, count, and do simple arithmetic (Geary, 1995).

**Secondary.** Biologically secondary, academic development involves the modification of primary brain and cognitive systems for the creation of culturally specific competencies (Geary, 1995). Some of these competencies are more similar to primary abilities than others. Spelke (2000) proposed that learning the Arabic numeral system involves integrating primary knowledge of small numbers and counting with the analog magnitude system. In this way, the primary understanding of small sets and the understanding that successive counts increase quantity by one can be extended beyond the small number sizes of the primary system. Siegler's and Opfer's (2003) research on children's number-line estimation is consistent with this proposal. Recall, Figure 18.2 shows children's number-line estimates that conform to predictions of the analog magnitude system (bottom) and estimates that conform to the Arabic, mathematical system (top). With experience the numerical ranges (e.g., 1 to 100 or 1 to 1,000), children's representation of the number line based on the biologically primary analog-magnitude system is gradually transformed into the secondary mathematical representation (Siegler & Booth, 2004).

Other features of academic mathematics, such as the base-10 system, are more remote from the supporting primary systems (Geary, 2002). Competency in base-10 arithmetic requires a conceptual understanding of the mathematical number line, and an ability to decompose this system into sets of ten and then to organize these sets into clusters of 100 (10, 20, 30 . . .), 1,000, and so forth. Whereas an implicit understanding of number sets is likely to be primary knowledge, the creation of sets around 10 and the superordinate organization of these sets is not. This conceptual knowledge must also be mapped onto a number-word system (McCloskey, Sokol, & Goodman, 1986), and integrated with school-taught procedures for solving complex arithmetic problems (Fuson & Kwon, 1992a). The development of base-10 knowledge thus requires the extension of primary number knowledge to very large numbers, the organization of these number representations in ways that differ from primary knowledge, and the learning of procedural rules for applying this knowledge to the secondary domain of

complex, mathematical arithmetic (e.g., to solve  $234 + 697$ ). In other words, multiple primary systems must be modified and integrated during children's learning of base-10 arithmetic.

Little is known about the mechanisms involved in modifying primary systems for learning secondary mathematics, but research on learning in other evolutionarily novel domains suggests the explicit attentional and inhibitory control mechanisms of the working memory system shown in Figure 18.3 are crucial (e.g., Ackerman & Cianciolo, 2002). It is known that some secondary mathematical learning, such as arithmetic facts or simple arithmetical relations, can occur implicitly but only with attentional focus and the repeated presentation of the same or similar arithmetical patterns (e.g., Siegler & Stern, 1998). In theory, the implicitly represented facts, concepts, and procedures (right-hand side of Figure 18.3) that result from this repetition will only emerge through modification underlying primary systems. There is no inherent understanding that  $9 + 7 = 16$ , but the repeated solving of this problem results in the development of a language-based (primary system) declarative fact (Dehaene & Cohen, 1991).

The explicit and conscious representation of information in working memory and through this the ability to make inferences about school-taught mathematics appears to require the synchronized activity of the dorsolateral prefrontal cortex and the activity of the brain regions that implicitly represent goal-related facts, concepts, or procedures (Dehaene & Naccache, 2001; Posner, 1994). The details of how this might occur as related to academic learning are presented elsewhere (Geary, 2005), but there are several basic points. First, repetition of mathematical problems in school should result in the formation of implicitly represented facts, concepts, or procedures in long-term memory (e.g., Siegler & Stern, 1998). This knowledge will influence problem solving and remain implicit unless the activity of the underlying brain regions becomes synchronized with activity of the dorsolateral prefrontal cortex, as will often happen when attention is focused on achieving a specific goal. When this occurs, the fact, concept, or procedural pattern will "pop" into conscious awareness, and thus be made available for drawing inferences about the information, forming metacognitive knowledge, and so forth. Second, an explicit representation of goal-related features in working memory (e.g., through direct instruction) should, in theory, result in a top-down influence on the pattern of activation and inhibi-

tion of brain regions that might support problem solving. These top-down processes may bias which implicitly represented facts, concepts, or procedures are used to approach the problem and may direct changes in the associated implicit knowledge.

In this view, change occurs through the repeated processing of a mathematical problem or pattern. The result is the repeated activation of a set of brain regions that process the pattern and through competition and selection at the level of corresponding neuronal groups (Edelman, 1987) form implicit biologically secondary facts, concepts, or procedures. Developmental change can occur with the maturation of the brain systems (e.g., dorsolateral prefrontal cortex) that support the explicit, working memory mechanisms shown in Figure 18.3. These regions appear to mature slowly through childhood and early adolescence (Giedd et al., 1999; Welsh & Pennington, 1988), and result in an improved ability to focus attention on goal-related problem solving, inhibit irrelevant information from interfering with problem solving, make explicit inferences about the goal and problem-solving processes, and form metacognitive knowledge.

## CONCLUSIONS AND FUTURE DIRECTIONS

During the past 2 decades and especially in recent years, the study of children's mathematical development has emerged as a vibrant and exciting area. Although there is much to be learned, we now have considerable knowledge of infants', preschoolers' and young children's understanding of number, counting, arithmetic, and some aspects of more complex arithmetical and algebraic problem solving (Brannon & Van de Walle, 2001; Briars & Larkin, 1984; Siegler, 1996; Starkey, 1992; Wynn, 1992a). As a sign of a maturing field, the empirical findings in most of these areas are no longer contested, but the underlying cognitive mechanisms and the mechanisms of developmental change are the foci of vigorous theoretical debate (Cohen & Marks, 2002; Newcombe, 2002). Much of the debate centers on the extent to which the most basic quantitative abilities emerge from inherent and potentially evolved brain and cognitive systems that are designed to attend to and process numerical features of the environment (Gallistel & Gelman, 1992; Wynn, 1995), or whether brain and cognitive systems designed for other functions (e.g., object identification) are the source of these abilities (Mix et al., 1997).

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Future research in this field will undoubtedly be focused on addressing this issue. In fact, recent behavioral (Brannon, 2002; Kobayashi et al., 2004) and brain-imaging (Pinel et al., 2004) studies have used sophisticated techniques to determine if the cognitive systems that support quantitative judgments are uniquely numerical, and whether the underlying brain regions are uniquely designed to process numerical information. The overall results are mixed, and thus a firm conclusion must await further studies. An equally important area of current debate and future study concerns the mechanisms that underlie children's cognitive development in the domain of mathematics and more generally (Siegler, 1996). We now have some understanding of the cognitive mechanisms (e.g., working memory, attention) that contribute to developmental change (Shrager & Siegler, 1998), and we are beginning to understand the operation of the supporting brain systems (Geary, 2005), but there is much to be learned about how these systems operate and change from one area of mathematics to another. A final direction is to apply knowledge derived from the scientific study of children's mathematical development to the study of how children learn mathematics in school.

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