Lecture 2. Intensity Transformation for Image Enhancement

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**General Idea**

- **Intensity transformation**
  - Also called point operation
  - Zero-memory operation

- Often, map a given gray level to another level

\[ s = T(r) \]

- Typically, \( T(.) \) is monotonically increasing (but not always)
Intensity Histogram

- Example
  a 4x4, 4bits/pixel image

<table>
<thead>
<tr>
<th>k</th>
<th>0</th>
<th>1</th>
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</tr>
</thead>
<tbody>
<tr>
<td>H(k)</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
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</tr>
</tbody>
</table>

```plaintext
1  8  6  6
6  3 11  8
8  8  9 10
9 10 10  7
```
Intensity Histogram (cont.)

From Gonzalez & Woods
Intensity Histogram (cont.)

From Gonzalez & Woods
Historgram Demo using OPENCV

- Demo from opencv: demhist.exe
  - This demo shows you the meaning of bright and contrast.
Fixed Intensity Transformation
Basic Transformations

Negative:
\[ s = L - 1 - r \]

Log:
\[ s = c \log(1 + r) \]

Inverse Log:
\[ s = e^{cr} - 1 \]

Power-law:
\[ s = cr^\gamma \]

.......
Negative Transformation

\[ s = L - 1 - r \]

FIGURE 3.4
(a) Original digital mammogram. (b) Negative image obtained using the negative transformation in Eq. (3.2-1). (Courtesy of G.E. Medical Systems.)
Log Transformation

\[ s = c \log(1 + r) \]

FIGURE 3.5
(a) Fourier spectrum. (b) Result of applying the log transformation given in Eq. (3.2-2) with \( c = 1 \).
Power-law (Gamma) Transformation

\[ s = cr^\gamma \]

**FIGURE 3.6** Plots of the equation \( s = cr^\gamma \) for various values of \( \gamma \) (\( c = 1 \) in all cases).

From Gonzalez & Woods
Power-law (Gamma) Transformation

\[ s = cr^\gamma \]

From Gonzalez & Woods
Power-law (Gamma) Transformation

\[ s = cr^\gamma \]

**FIGURE 3.9**
(a) Aerial image. (b)–(d) Results of applying the transformation in Eq. (3.2-3) with \( c = 1 \) and \( \gamma = 3.0, 4.0, \text{ and } 5.0 \), respectively. (Original image for this example courtesy of NASA.)
Thresholding

\[ s = \begin{cases} 
0 & \text{if } r \leq m \\
cl & \text{if } r > m 
\end{cases} \]

\[ m : \text{threshold} \]
Example: Fixed Intensity Transformation

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A 4x4, 4bits/pixel image passes through an intensity transformation

\[ s = T(r) = \text{round}\left(\frac{1}{15} r^2\right) \]

\begin{align*}
1 & \to \text{round}(0.0667) = 0; \\
3 & \to \text{round}(0.6) = 1; \\
6 & \to \text{round}(2.4) = 2; \\
7 & \to \text{round}(3.2667) = 3; \\
8 & \to \text{round}(4.2667) = 4; \\
9 & \to \text{round}(5.4) = 5; \\
10 & \to \text{round}(6.6667) = 7; \\
11 & \to \text{round}(8.0667) = 8;
\end{align*}

The resulting image is:

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<tbody>
<tr>
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</table>
# Resulting Histogram Change

<table>
<thead>
<tr>
<th>1</th>
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<tr>
<th>0</th>
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<td>7</td>
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<td>3</td>
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</tbody>
</table>

![Histogram Diagram]
Contrast Stretch
Contrast Stretch

General Idea: Make Best Use of the Dynamic Range

From Gonzalez & Woods
Contrast Stretch (cont.)

General form:
\[
s = \begin{cases} 
\frac{s_1}{r_1} \cdot r & 0 \leq r < r_1 \\
\frac{s_2 - s_1}{r_2 - r_1} \cdot r + \frac{s_1 r_2 - s_2 r_1}{r_2 - r_1} & r_1 \leq r \leq r_2 \\
\frac{2^B - 1 - s_2}{2^B - 1 - r_2} \cdot r + (2^B - 1) \cdot \frac{s_2 - r_2}{2^B - 1 - r_2} & r_2 < r \leq 2^B - 1 
\end{cases}
\]

Special case \(\rightarrow\) Full-scale contrast stretch:
\[
\begin{align*}
r_1 &= r_{\min} & s_1 &= 0 \\
r_2 &= r_{\max} & s_2 &= 2^B - 1
\end{align*}
\]

Typically used:
\[
s = \text{round} \left( (2^B - 1) \cdot \frac{r - r_{\min}}{r_{\max} - r_{\min}} \right)
\]
Contrast Stretch (cont.)

- If more interested in the image intensity values within \([r_1, r_2]\) than any other range, stretch the intensity values within \([r_1, r_2]\) properly.

- If \(r_1\) and \(r_2\) are not known, a full-scale contrast stretch is normally used to fully expand the dynamic range as allowed by the allocated number of bits.
Example: Full-Scale Contrast Stretch

- Full-scale contrast stretch of a 4x4, 4 bits/pixel image

- Find \( r_{\text{min}} = 4 \quad r_{\text{max}} = 11 \quad 2^B - 1 = 15 \)

\[
s = \text{round} \left( (2^B - 1) \cdot \frac{r - r_{\text{min}}}{r_{\text{max}} - r_{\text{min}}} \right) = \text{round} \left( 15 \cdot \frac{r - 4}{11 - 4} \right) = \text{round} \left( \frac{15}{7} (r - 4) \right)
\]

4 \( \rightarrow \) round(0) = 0;
6 \( \rightarrow \) round(4.29) = 4;
7 \( \rightarrow \) round(6.43) = 6;
8 \( \rightarrow \) round(8.57) = 9;
9 \( \rightarrow \) round(10.71) = 11;
10 \( \rightarrow \) round(12.86) = 13;
11 \( \rightarrow \) round(15) = 15;

The resulting image is:
### Resulting Histogram Change

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<tbody>
<tr>
<td>0</td>
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<td>4</td>
<td>4</td>
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<td>4</td>
<td>0</td>
<td>15</td>
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<td>13</td>
</tr>
<tr>
<td>9</td>
<td>15</td>
<td>13</td>
<td>6</td>
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</tbody>
</table>
Histogram Equalization
Histogram Equalization

The left images: dark light; the right images: results of histogram equalization
Histogram Equalization (cont.)

The left images: low contrast, high contrast; the right images: results of histogram equalization.
An Example to Illustrate why histogram equalization is needed

- A 4x4, 4bits/pixel image

```
2 8 9 9
2 3 10 9
8 3 3 11
8 3 10 11
```

- First try: full-scale contrast stretch \( r_{\text{min}} = 2 \quad r_{\text{max}} = 11 \)

\[
s = \text{round}\left((2^B - 1) \cdot \frac{r - r_{\text{min}}}{r_{\text{max}} - r_{\text{min}}}\right) = \text{round}\left(15 \cdot \frac{r - 2}{11 - 2}\right) = \text{round}\left(\frac{5}{3}(r - 2)\right)
\]

2 \(\rightarrow\) round(0) = 0;
3 \(\rightarrow\) round(1.67) = 2;
8 \(\rightarrow\) round(10.00) = 10;
9 \(\rightarrow\) round(11.67) = 12;
10 \(\rightarrow\) round(13.33) = 13;
11 \(\rightarrow\) round(15) = 15;

The resulting image is:

```
0 10 12 12
0 2 13 12
10 2 2 15
10 2 13 15
```
### Resulting Histogram Change

<table>
<thead>
<tr>
<th>original</th>
<th>full-scale contrast stretch</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="image1.png" alt="Histogram" /></td>
<td><img src="image2.png" alt="Histogram" /></td>
</tr>
</tbody>
</table>

- **Original Histogram**
  - Values: 2, 8, 9, 9
  - Values: 2, 3, 10, 9
  - Values: 8, 3, 3, 11
  - Values: 8, 3, 10, 11

- **Full-Scale Contrast Stretch**
  - Values: 0, 10, 12, 12
  - Values: 0, 2, 13, 12
  - Values: 10, 2, 2, 15
  - Values: 10, 2, 13, 15

**Notes:**
- **Big Gap** indicates a significant change in the histogram values.
- The full-scale contrast stretch adjusts the histogram to fit the entire scale.
Desired properties

- A fixed monotonic intensity transformation transforms an original image into an image, which looks alike the original one.

- We would love to find such a transformation, which expands the full usable dynamic range as specified by the number of bits.

- We would love to see that for each intensity value there always exist pixels taking on that value or at least there is no large gaps appearing in the histogram of the transformed image.
Cumulative Histogram

<table>
<thead>
<tr>
<th>k</th>
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<th>1</th>
<th>2</th>
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<th>12</th>
<th>13</th>
<th>14</th>
<th>15</th>
</tr>
</thead>
<tbody>
<tr>
<td>H(k)</td>
<td>0</td>
<td>0</td>
<td>2</td>
<td>4</td>
<td>0</td>
<td>0</td>
<td>0</td>
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<td>3</td>
<td>2</td>
<td>2</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Q(k)</td>
<td>0</td>
<td>0</td>
<td>2</td>
<td>6</td>
<td>6</td>
<td>6</td>
<td>6</td>
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</tr>
</tbody>
</table>

H(k) and Q(k) histograms are shown.
### Intermediate Image

<table>
<thead>
<tr>
<th>k</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
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<th>12</th>
<th>13</th>
<th>14</th>
<th>15</th>
</tr>
</thead>
<tbody>
<tr>
<td>H(k)</td>
<td>0</td>
<td>0</td>
<td>2</td>
<td>4</td>
<td>0</td>
<td>0</td>
<td>0</td>
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<td>3</td>
<td>3</td>
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<td>2</td>
<td>0</td>
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<tr>
<td>Q(k)</td>
<td>0</td>
<td>0</td>
<td>2</td>
<td>6</td>
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<td>6</td>
<td>6</td>
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</tbody>
</table>

### Original Image

\[
\begin{array}{cccc}
2 & 8 & 9 & 9 \\
2 & 3 & 10 & 9 \\
8 & 3 & 3 & 11 \\
8 & 3 & 10 & 11 \\
\end{array}
\]

### Intermediate Image

\[
\begin{array}{cccc}
2 & 9 & 12 & 12 \\
2 & 6 & 14 & 12 \\
9 & 6 & 6 & 16 \\
9 & 6 & 14 & 16 \\
\end{array}
\]
Full-Scale Contrast Stretch of Intermediate Image

intermediate image

\[
\begin{array}{ccc}
2 & 9 & 12 12 \\
2 & 6 & 14 12 \\
9 & 6 & 6 16 \\
9 & 6 & 14 16 \\
\end{array}
\]

\[ r_{\text{min}} = 2 \quad r_{\text{max}} = 16 \]

\[
s = \text{round}\left(2^B - 1 \cdot \frac{r - r_{\text{min}}}{r_{\text{max}} - r_{\text{min}}} \right) = \text{round}\left(15 \cdot \frac{r - 2}{16 - 2} \right) = \text{round}\left(\frac{15}{14} (r - 2) \right)
\]

2 \rightarrow \text{round}(0) = 0;
6 \rightarrow \text{round}(4.29) = 4;
9 \rightarrow \text{round}(7.50) = 8;
12 \rightarrow \text{round}(10.71) = 11;
14 \rightarrow \text{round}(12.86) = 13;
16 \rightarrow \text{round}(15) = 15;

final result: histogram equalized image

\[
\begin{array}{ccc}
0 & 8 & 11 11 \\
0 & 4 & 13 11 \\
8 & 4 & 4 15 \\
8 & 4 & 13 15 \\
\end{array}
\]
Histogram Comparison

original  direct full-scale contrast stretch  histogram-equalized

<table>
<thead>
<tr>
<th>4</th>
<th>8</th>
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<table>
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<tbody>
<tr>
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</table>

more equalized
Summary of the Histogram Equalization Algorithm

original image

histogram $H(k)$

cumulative histogram $Q(k)$

intermediate image

full-scale contrast stretch

histogram-equalized image
The Justification of histogram equalization

- A proof

- We focus our attention on transforms of the form: \( y = T(x), \quad 0 \leq x \leq L - 1 \), such that:
  - \( T(x) \) is strictly monotonically increasing function in the interval \( 0 \leq x \leq L - 1 \).
  - \( 0 \leq T(x) \leq L - 1 \), for \( 0 \leq x \leq L - 1 \). Therefore \( x = T^{-1}(y), \quad 0 \leq y \leq L - 1 \).

- The intensity level in an image may be viewed as a random variable in \([0, L - 1]\).

- We use \( f_X(x) \) and \( f_Y(y) \) to denote the PDFs of \( X \) and \( Y \), i.e. the normalized histograms.
A fundamental result from basic probability theory is that if \( f_X(x) \) and \( T(x) \) are known, and \( T(x) \) is differentiable and strictly monotonic over the range of interests, then the PDF of the transformed (mapped) random variable \( Y \) is given as:

\[
f_Y(y) = \begin{cases} 
    f_X(T^{-1}(y)) \left| \frac{d}{dy} T^{-1}(y) \right|, & \text{if } y = T(x) \text{ for some } x \\
    0, & \text{if } y \neq T(x) \text{ for all } x
\end{cases}
\]

(P221 of *A First Course in Probability*, by Sheldon Ross, 8th edition.)
Proof (Cont.)

\[ f_Y(y) = \begin{cases} 
  f_X(T^{-1}(y)) \left| \frac{d}{dy} T^{-1}(y) \right| , & \text{if } y = T(x) \text{ for some } x \\
  0 , & \text{if } y \neq T(x) \text{ for all } x 
\end{cases} \]

Proof: Suppose that \( y = T(x) \) for some \( x \). Then with \( Y = T(X) \),

\[
F_Y(y) = P\{T(X) \leq y\} \\
= P\{X \leq T^{-1}(y)\} \\
= F_X(T^{-1}(y))
\]

Differentiation gives:

\[
f_Y(y) = f_X(T^{-1}(y)) \left| \frac{d}{dy} T^{-1}(y) \right|
\]

Since \( T^{-1}(y) \) is non-decreasing, so its derivative is non-negative.
Proof (Cont.)

When \( y \neq T(x) \) for any \( x \), then \( F_Y(y) \) is either 0 or 1, and in either case \( f_Y(y) = 0 \).

Therefore, we proved the PDF of the mapped random variable is:

\[
f_Y(y) = \begin{cases} 
    f_X(T^{-1}(y)) \left| \frac{d}{dy} T^{-1}(y) \right|, & \text{if } y = T(x) \text{ for some } x \\
    0, & \text{if } y \neq T(x) \text{ for all } x
\end{cases}
\]

(P221 of *A First Course in Probability*, by Sheldon Ross, 8th edition.)
The Justification of histogram equalization

- A transformation function of particular importance in image processing has the form
  \[ y = T(x) = (L - 1) \int_{0}^{x} f_X(w) \, dw \]

- For this specific transform, we have:
  \[ \frac{dy}{dx} = \frac{dT(x)}{dx} = (L - 1)f_X(x) \]

- From the proved equation of PDF of the mapped random variable, we have (Note: \( x = T^{-1}(y) \)):
  \[ f_Y(y) = f_X(T^{-1}(y)) \frac{d}{dy} T^{-1}(y) \]
  \[ = f_X(x) \left| \frac{dx}{dy} \right| = \frac{f_X(x)}{(L-1)f_X(x)} = \frac{1}{L-1} \]
The Justification of histogram equalization

Therefore:

\[ f_Y(y) = f_X(T^{-1}(y)) \left| \frac{d}{dy} T^{-1}(y) \right| \]

\[ = f_X(x) \left| \frac{dx}{dy} \right| = \frac{f_X(x)}{\left| \frac{dy}{dx} \right|} = \frac{f_X(x)}{(L-1)f_X(x)} = \frac{1}{L-1} \]
Connection between histogram equalization and random number generation

- Image equalization is inherently an integral mapping: 
  \[ y = T(x) = (L - 1) \int_0^x f_X(w)\,dw \], which map *arbitrarily distributed* random variable \( X \) into a *uniformly distributed* random variable \( Y \).

- We can the invert operation to generate arbitrarily distributed random variable \( X \) from a uniform random variable generator \( Y \).

Suppose \( f_X(x) = 0 \) for \( x < a \),

\[
y = T(x) = (L - 1) \int_a^x f_X(w)\,dw
\]

\[
\therefore \frac{y}{L - 1} = F_X(x) - F_X(a)
\]

\[
\therefore x = F^{-1}\left(\frac{y}{L - 1} + F_X(a)\right)
\]
Arbitrary RV from Uniform RV

- A Matlab function (SampleFrmP.m)

```matlab
function [Idx]=sampleFrmP(Pmf)
% sample a random variable of arbitrary distribution distributed as Pmf.
Plen=length(Pmf);
cdf=zeros(Plen,1);
cdf(1)=Pmf(1);
C=Pmf(1); % the normalization constant
for k=2:Plen
    cdf(k)=cdf(k-1)+Pmf(k);
    C= C+Pmf(k);
end
x=rand(1)=C;
for k=1: Plen
    if x< cdf(k)
        Idx=k;
        break;
    end
end
return;
```

Here generate r.v. Idx distributed as the distribution specified by Pmf.
Histogram specification

- Histogram Specification is a generalized version of histogram equalization.

- An equalized image has an equal number of pixels at all brightness levels, resulting in a straight horizontal line on the histogram graph.

- When you specify a histogram, you actually define the desired shape of the histogram, and a nonlinear stretch operation is applied to force the image histogram to have that shape.

source:
http://www.cyanogen.com/help/maximdl/HID_PROC_HISTSPECIF.htm
Histogram specification

- Here we want to convert the image so that it has a particular histogram that can be arbitrarily specified.
- Such a mapping function can be found in three steps:
  - Equalize the histogram of the input image
  - Equalize the specified histogram
  - Relate the two equalized histograms

source:
http://fourier.eng.hmc.edu/e161/lectures/contrast_transform/node3.html