Bayesian and frequentist tests of sign equality and other nonlinear inequalities

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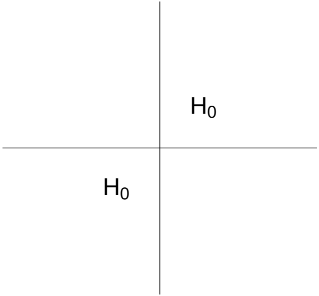
Outline

Setup (1/2)

- Unknown: $\theta_1 \in \mathbb{R}$, $\theta_2 \in \mathbb{R}$
- $H_0 : \theta_1\theta_2 \geq 0,\ H_1 : \theta_1\theta_2 < 0$



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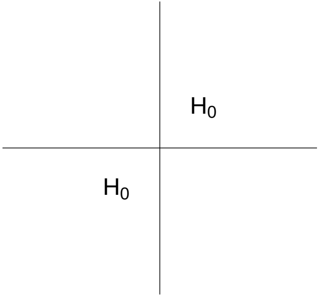

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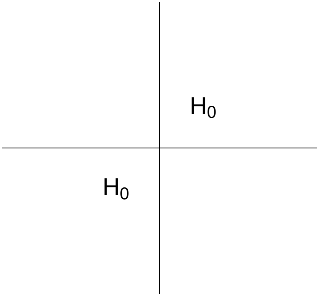
- Ex (IV): $y = x\beta + u$, $x = z\pi + v$; $y = z\pi\beta + \eta$, $\theta_1 = \pi$, $\theta_2 = \pi\beta$; $H_0 : \beta \geq 0 \Leftrightarrow H_0 : \theta_1\theta_2 \geq 0$ since $\pi^2 \geq 0$



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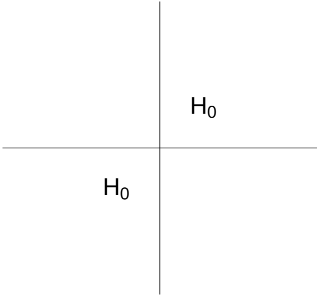
- Ex: stability over time, θ_t, $t \in \{1, 2\}$



Setup (2/2)

- Local-to-zero asy: $\theta_1 = d_1/\sqrt{n}$, $\theta_2 = d_2/\sqrt{n}$,

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- Limit experiment: single draw of

$$\begin{pmatrix}\hat{t}_1 \\ \hat{t}_2\end{pmatrix} \sim N\left(\begin{pmatrix}d_1 \\ d_2\end{pmatrix}, \begin{pmatrix}1 & \rho \\ \rho & 1\end{pmatrix}\right),$$

known ρ, unknown d_1, d_2; $\hat{t} = \sqrt{n}\hat{\theta}/\hat{\sigma}$.

Likelihood ratio (LR) test: derivation

- Let $\rho = 0$ for now; let $\alpha = 0.05$
- LR: Euclidean distance from H_0 (iso-LR: concentric circles)

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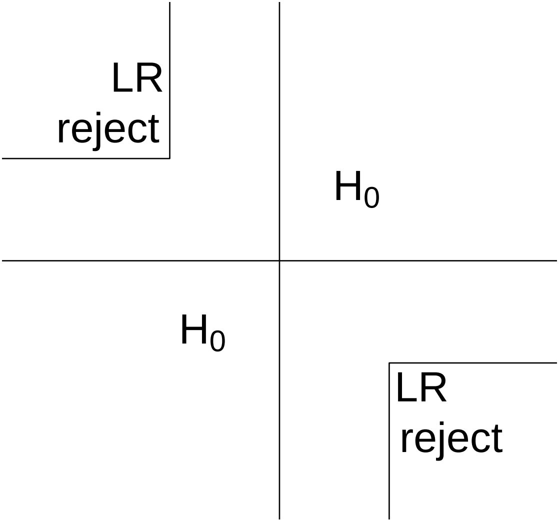
- Very conservative near origin

LR test: rejection region



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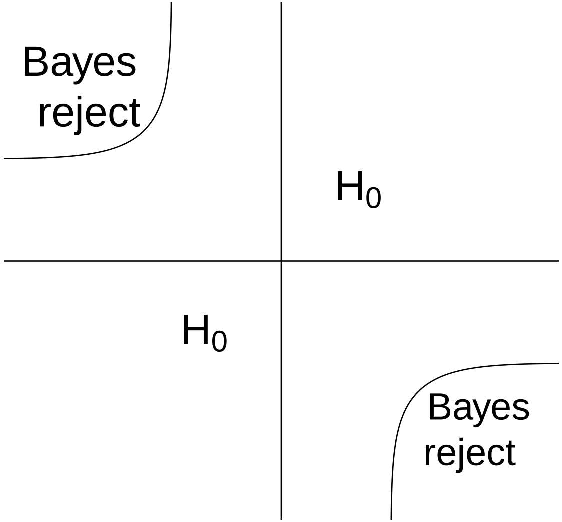
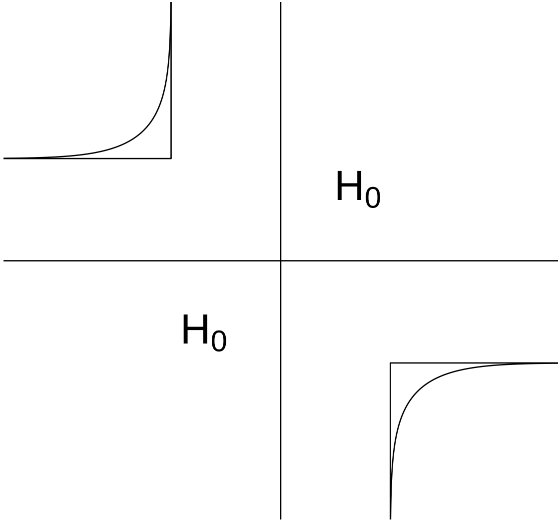
- If $P(H_0) = \alpha$, then $\mathbb{E}(L \mid \text{rej}) = \mathbb{E}(L \mid \text{acc}) = \alpha(1-\alpha)$
- If $P(H_0) < \alpha$, should reject; if $P(H_0) > \alpha$, accept

Bayesian test: rejection region



LR vs. Bayesian rejection regions



LR vs. Bayesian rejection regions

- I was surprised (just me?)
- Kline (2011): $H_0 : \mu \geq 0$, $\mu \in \mathbb{R}^k$, can have $P(H_0) \approx 0$ and frequentist p-value≈ 1
- Moon and Schorfheide (2012, Cor. 1): for set-identified parameter, Bayesian credible interval is strict subset of frequentist CI
- Bayes rule vs. minimax risk

1D: standard one-sided testing

- Sampling dist: $X \sim N(\mu, 1)$; posterior: $\mu \sim N(X, 1)$
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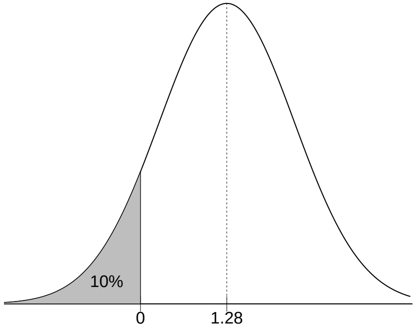
- $H_0 : \mu \leq 0$, $H_1 : \mu > 0$, $\alpha = 0.1$
- Bayesian: $P(H_0 \mid X = 1.28) = \alpha$; reject if $X > 1.28$, otherwise accept



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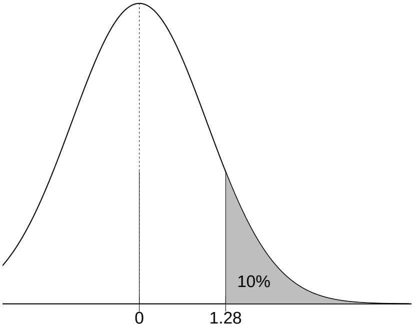

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- "Unbiased": rejection probability (RP) given any $\mu \leq 0$ is $\leq$ RP given any $\mu > 0$
- Minimax risk (expected loss) decision rule:
  - Same loss as before: $1 - \alpha$ for type I, α for II
  - Let $\mathrm{RP}(\mu)$ be RP given μ, continuous in μ
  - Given $\mu \leq 0$, risk is $(1 - \alpha)\mathrm{RP}(\mu)$
  - Given $\mu > 0$, risk is $\alpha(1 - \mathrm{RP}(\mu))$
  - $\mathrm{RP}(\mu = 0) = \mathrm{RP}(\mu \to 0+)$, so min max risk by $\mathrm{RP}(0) = \alpha$

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- $X \sim N(\mu, 1)$, $\mu \sim N(X, 1)$
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- Small h: similar
- Frequentist testing is "more conservative" than Bayesian

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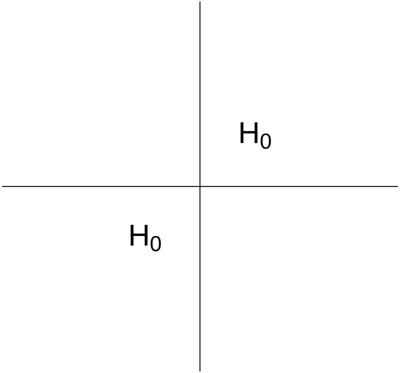
Sign equality with $\rho = -1$

- $H_0 : d_1 d_2 \geq 0$, $H_1 : d_1 d_2 < 0$
- If $\rho = -1$, then observing $(\hat{t}_1, \hat{t}_2)$ implies $\hat{t}_1 - d_1 = -(\hat{t}_2 - d_2)$, so $d_2 = (\hat{t}_1 + \hat{t}_2) - d_1$



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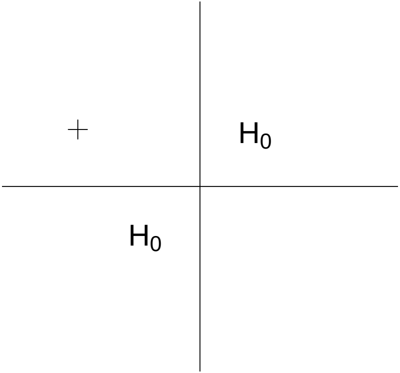

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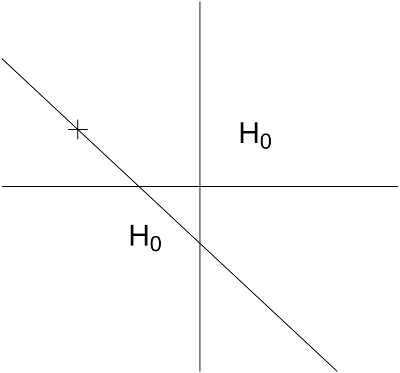

- Reduced to single unknown parameter, convex H_0

Sign equality with $\rho = 1$

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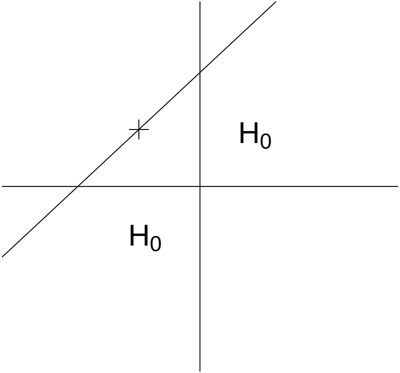

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General result

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- (Theorem 5) If Θ_0 is a convex set, then frequentist testing is "more conservative": the frequentist RP of the Bayesian test given any θ on the boundary of Θ_0 is at least α
- Corollary: non-convexity of Θ_0 is a necessary but not sufficient condition for Bayesian testing to be more conservative (as seen in 1D)

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- Corollary: non-convexity of Θ_0 is a necessary but not sufficient condition for Bayesian testing to be more conservative (as seen in 1D)
- (Corollary 6) Linear inequalities produce convex Θ_0, so frequentist testing is more conservative; ex: $H_0 : \theta \geq 0$

Unbiased moment inequality tests

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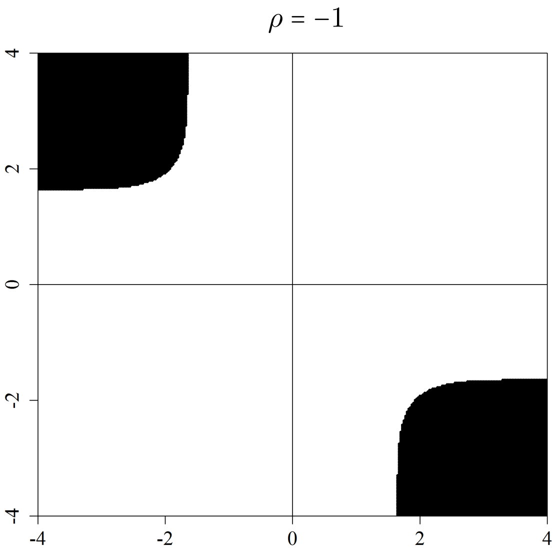
- Andrews (2012): "Similar-on-the-boundary tests for moment inequalities exist, but have poor power" for $H_0 : \mu \geq 0$ (vector)
- Argument doesn't apply here since $\{\theta : \theta_1\theta_2 = 0\}$ is entirely on boundary of H_0 for sign equality
- But: applies to sign equality in higher dimensions

Unbiased sign equality test

- How to derive? Chiburis (2008), Moreira and Moreira (2013): frame as linear program, max weighted avg power (WAP) s.t. (approx) unbiasedness. See also Berger (1989, §6) for $\rho = 0$
- Pro: unbiased; minimax risk
- Pro: power dominates LR over H_1 (but risk fn doesn't dominate over H_0)
- Con: computation; LR and Bayesian tests much simpler/faster
- Con: unbiased test lacks a type of "rejection monotonicity," and (for larger $\rho > 0$) can reject even if $\hat{t}$ satisfies H_0; see Perlman and Wu (1999) for more detailed criticism

Rejection regions for various ρ, $\alpha = 0.05$



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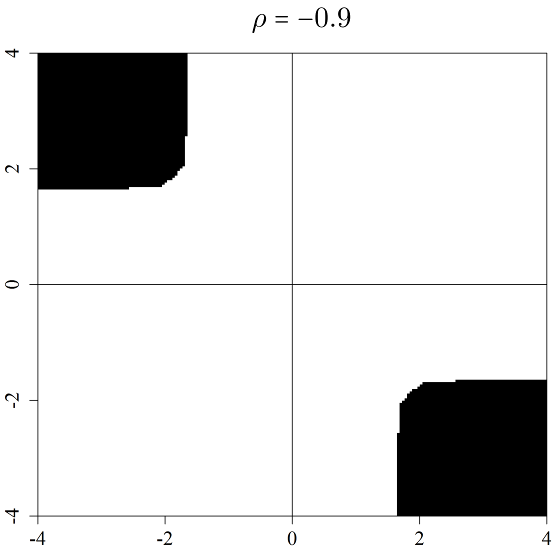

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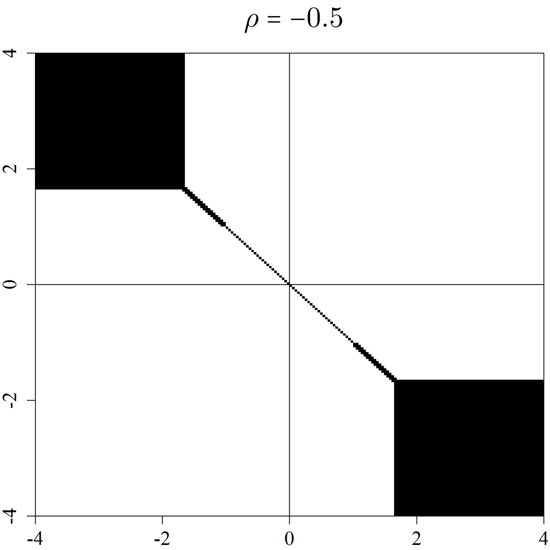

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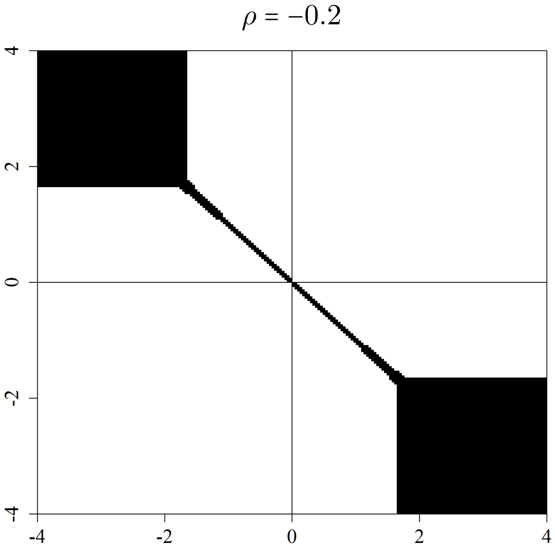
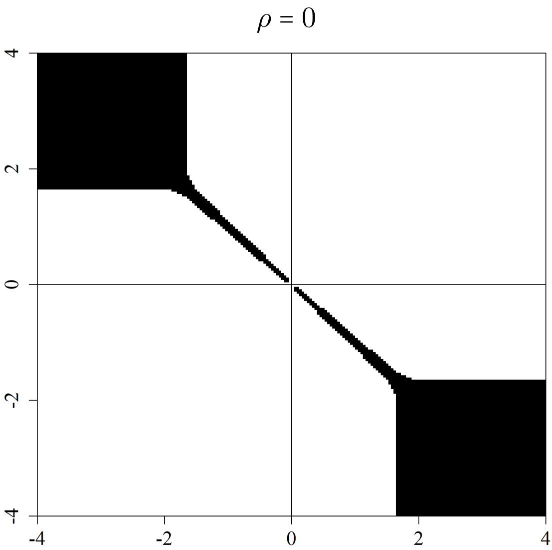
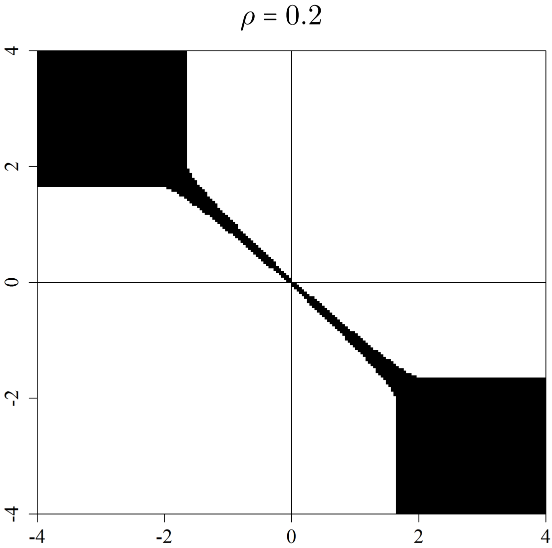


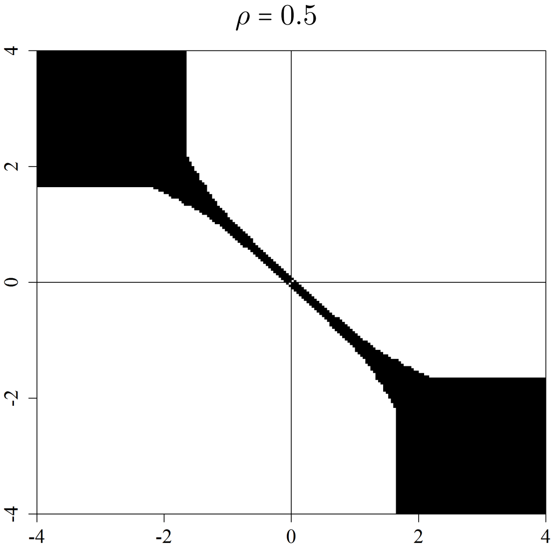
Rejection regions for various ρ, $\alpha = 0.05$



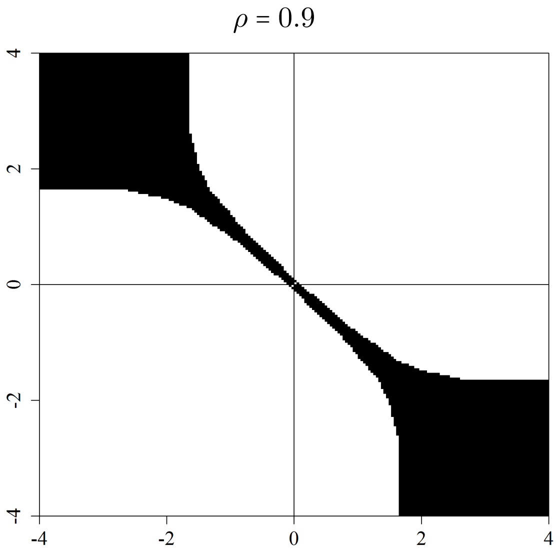
Rejection regions for various ρ, $\alpha = 0.05$



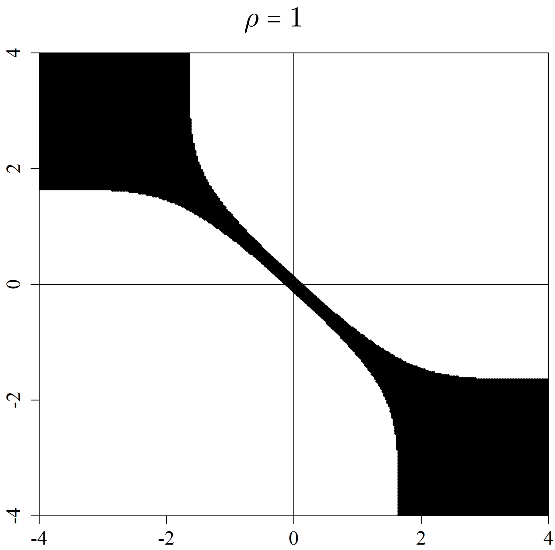
Rejection regions for various ρ, $\alpha = 0.05$



Rejection regions for various ρ, $\alpha = 0.05$



Rejection regions for various ρ, $\alpha = 0.05$



Recap

- Frequentist testing is "more conservative" than Bayesian testing with convex Θ_0, given equivalent asymptotic normal sampling and posterior distributions
- Linear inequalities always define a convex Θ_0
- Sign equality Θ_0 is non-convex; which is more conservative depends on ρ
- Unlike for $H_0 : \theta \geq 0$, unbiased tests with good power exist for $H_0 : \theta_1\theta_2 \geq 0$ (but not higher dimensions)
- But, unbiasedness comes at a cost

Bayesian and frequentist tests of sign equality and other nonlinear inequalities

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MEG 2015

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