Smoothed IV quantile regression and quantile Euler equations

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Outline

Standard consumption Euler equation

- Expected utility maximization, $U(C) = C^{1-\gamma}/(1-\gamma)$:

$$0 = \mathbb{E}[\beta(1 + R_{t+1})(C_{t+1}/C_t)^{-\gamma} - 1 \mid \Omega_t],$$

  Ω_t: information set at time t
  R_{t+1}: real rate of return of asset
  C_t: real consumption at time t
  β: discount factor (e.g., $\beta = 0.99$)
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- Estimation: use variables in Ω_t as instruments (inflation, etc.); run GMM, or IV/2SLS after log-linearization
- Drawback: no separation of EIS ($1/\gamma$) and risk aversion (γ)
- Drawback: approximation error from log-linearization

Quantile Euler equation?

- Standard: $0 = \mathbb{E}[\beta(1 + R_{t+1})(C_{t+1}/C_t)^{-\gamma} - 1 \mid \Omega_t]$
- Replace $\mathbb{E}[\cdot \mid \Omega_t]$ with conditional τ-quantile $Q_\tau[\cdot \mid \Omega_t]$:

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- Grounded in decision theory? (next slide)
- Practical to estimate? (SIVQR)

Quantile Euler equation: decision theory

- Quantile utility maximization, static setting: Manski (1988), Chambers (2009), and Rostek (2010) (axiomatization)
- Dynamic setting: de Castro and Galvao (2017) show dynamic consistency and derive Euler equation

Quantile Euler equation: estimation

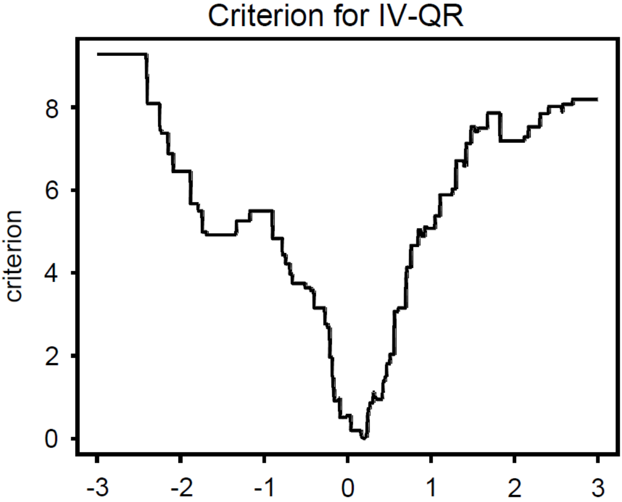
- Can write as $Q_\tau[\epsilon_{t+1} \mid \Omega_t] = 1$, $\epsilon_{t+1} \equiv \beta(1 + R_{t+1})(C_{t+1}/C_t)^{-\gamma}$
- Since $\ln(\cdot)$ is strictly increasing, $Q_\tau[\ln(W)] = \ln(Q_\tau[W])$
- In contrast, $\mathbb{E}[\ln(W)] \le \ln(\mathbb{E}[W])$ (Jensen's); approx error

IV quantile regression (IVQR)

- So we can just run IVQR; but. . .

IV quantile regression (IVQR)

Chernozhukov and Hong (2003), Figure 1(a)



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- Chernozhukov and Hansen (2006): iid, linear model, only 1 or 2 endogenous regressors (so can't have many interactions, polynomial terms, etc.)
- Other methods (also iid): compliers/LQTE (Abadie, Angrist, and Imbens, 2002), MCMC (Chernozhukov and Hong, 2003; Lancaster and Jun, 2010), triangular system (Lee, 2007, and others), very slow MIQP (Chen and Lee, 2017)

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- de Castro, Galvao, and Kaplan (2017): dependent data, nonlinear model; fast and robust computation, consistency and asymptotic normality
- New results underway from Xin Liu: smoothed two-step GMM, computation (code) and asymptotic theory

Outline

Smoothed IVQR (SIVQR): benefits

- Approach: smooth the moment conditions (estimating equations)
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- Initially: use just-identified system for numerical robustness; if over-identified, just take linear combination of instruments
- Benefit #1: computation is feasible, fast, scalable (many endogenous regressors), and numerically robust
- Benefit #2: often improves MSE (Kaplan and Sun, 2017)
- Benefit #3: important first step toward true IV quantile GMM (in progress by Xin Liu)

Smoothed QR (not IV): literature

- Horowitz (1998): smooths criterion fn instead of moments; Studentized bootstrap refinement
- Whang (2006): same moment smoothing used here, but for empirical likelihood QR; also in Otsu (2008)
- Fernandes, Guerre, and Horta (2017): kernel-smoothed QR criterion; FOC same as smoothed moments above
- MaCurdy and Hong (1999): original IVQR smoothing? (unpub'd notes)

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- Simplistic example: linear structural random coefficient model, $Y = \mathbf{X}'\beta(U)$, assume $\mathbf{X}'\beta(U)$ monotonic in unobserved $U \sim \mathrm{Unif}(0,1)$
- If Y is wage, U is "ability": $\mathbf{X}'\beta(0.5)$ traces out potential wage outcomes (given different $\mathbf{X}$) for individual with median ability $(\mathrm{P}(U \leq 0.5) = 0.5)$

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- If instrument vector $\mathbf{Z} \perp\!\!\!\perp U$, then $\mathrm{P}(Y \le \mathbf{X}'\beta(\tau) \mid \mathbf{Z}) = \mathrm{P}(U \le \tau \mid \mathbf{Z}) = \mathrm{P}(U \le \tau) = \tau$: a conditional quantile restriction on the observables Y, $\mathbf{X}$, and $\mathbf{Z}$, and the parameter vector $\beta(\tau)$

Alternative quantile models with endogeneity

- Triangular system: compared in Chernozhukov, Hansen, and Wüthrich (2017, §2.5); like Chesher (2003), Lee (2007), et al.
- LQTE (like LATE): compared in Melly and Wüthrich (2017, §§5–6); like Abadie et al. (2002), Frölich and Melly (2013), et al.

IVQR moment conditions

(linear IV)

$$\tau = \mathrm{P}(Y \le h(\mathbf{X}, \tau) \mid \mathbf{Z})$$

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$$\tau = \mathrm{P}(Y \le h(\mathbf{X},\tau) \mid \mathbf{Z}),\ h(\mathbf{X},\tau) = \mathbf{X}'\boldsymbol{\beta}_{0\tau},\ \mathrm{P}(\cdot) = \mathbb{E}(\mathbb{1}\{\cdot\}) \implies$$

$$\mathbf{0} = \mathbb{E}\Big[\mathbf{Z}\Big(\mathbb{1}\Big\{Y - \mathbf{X}'\boldsymbol{\beta}_{0\tau} \le 0\Big\} - \tau\Big)\Big]$$ (linear IVQR)

IVQR moment conditions

$$Y = h(\mathbf{X}) + U,\ \mathbb{E}(U \mid \mathbf{Z}) = 0,\ h(\mathbf{X}) = \mathbf{X}'\boldsymbol{\beta}_0 \implies$$

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Just run GMM?

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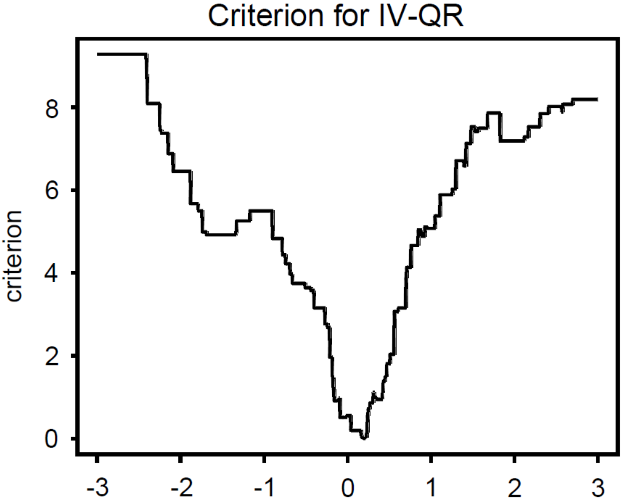
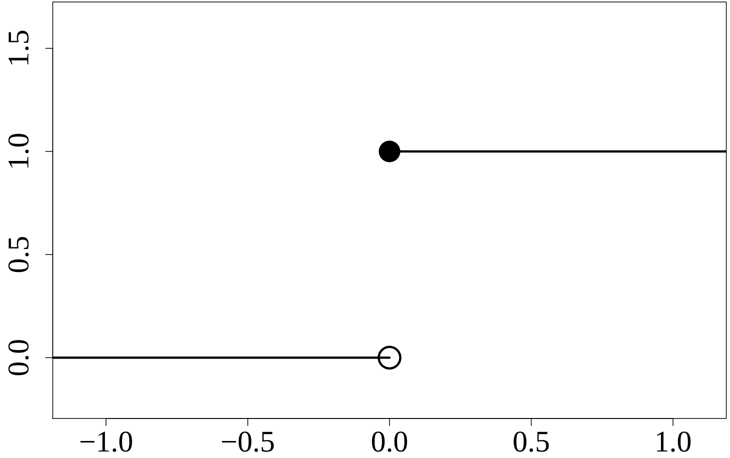


Smoothing the indicator function



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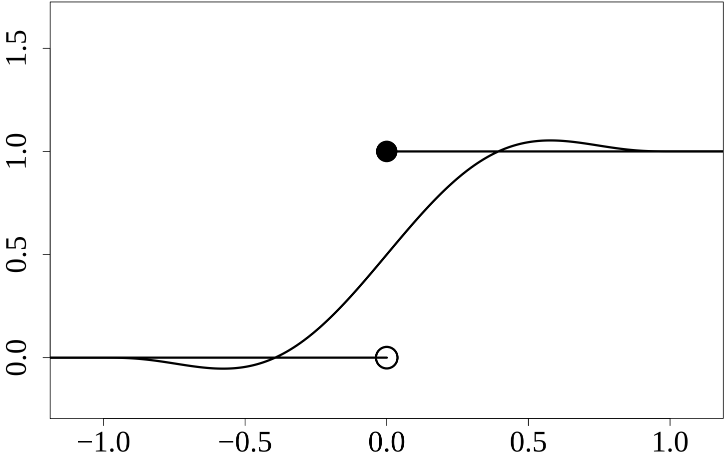

Connections: IV, Winsorized mean

$$\mathbf{0} = n^{-1} \sum_{i=1}^{n} \mathbf{Z}_i \left[\tilde{I}\left(\frac{\mathbf{X}_i' \hat{\boldsymbol{\beta}}_\tau - Y_i}{h_n} \right) - \tau \right]$$

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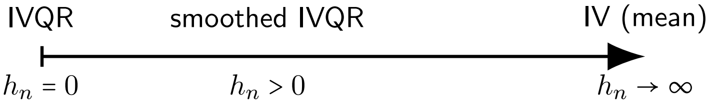

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- Can try to pick h_n to improve efficiency (like median vs. mean); maybe even better to explicitly average IVQR and IV (and QR) like Hansen (2017)?
- Special case: $X_i = Z_i = 1$, $\tilde{I}'(u) = \mathbb{1}\{-1 \le u \le 1\}/2$, $\tau = 0.5 \implies$ "Winsorized" mean (Huber, 1964, Ex. iii, p. 79)

Outline

1. Consumption Euler equations
2. Smoothed IV quantile regression (SIVQR)
3. Results
4. Conclusion

JTPA: context

- Abadie, Angrist, and Imbens (2002), 5102 adult men
- Randomized offer of services to individuals (Z_i), 62% uptake: $\mathrm{P}(D_i = 1 \mid Z_i = 1) = 0.62$.
- Other regressors: age, race, etc.
- Endogeneity from self-selection into treatment; OLS estimate twice as big as IV est
- Y_i: 30-month earnings (US dollars) in "after" period

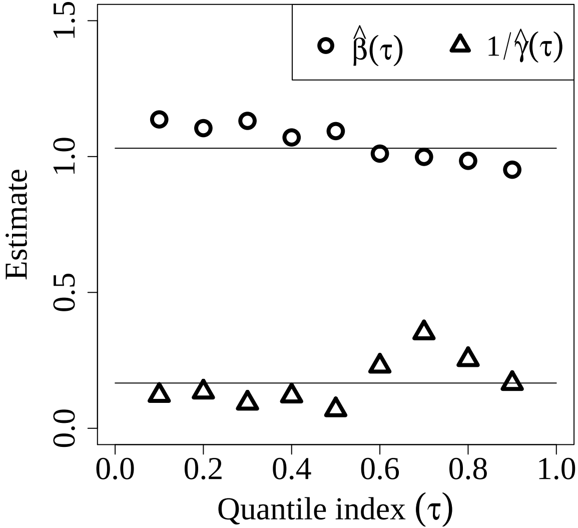
JTPA: results

		Quantile index τ				
Regressor	Method	0.15	0.25	0.50	0.75	0.85
Training	AAI	121	702	1544	3131	3378
Training	SEE ($\hat{h}$)	57	381	1080	2630	2744
Training	CH	−125	341	385	2557	3137
Training	tiny h	−129	500	381	2760	3114
Training	huge h	1579	1584	1593	1602	1607
Training	2SLS			1593		
Married	AAI	1564	3190	7683	9509	10,185
Married	SEE ($\hat{h}$)	1132	2357	7163	10,174	10,431
Married	CH	504	2396	7722	10,463	10,484
Married	tiny h	504	2358	7696	10,465	10,439
Married	huge h	6611	6624	6647	6670	6683
Married	2SLS			6647		

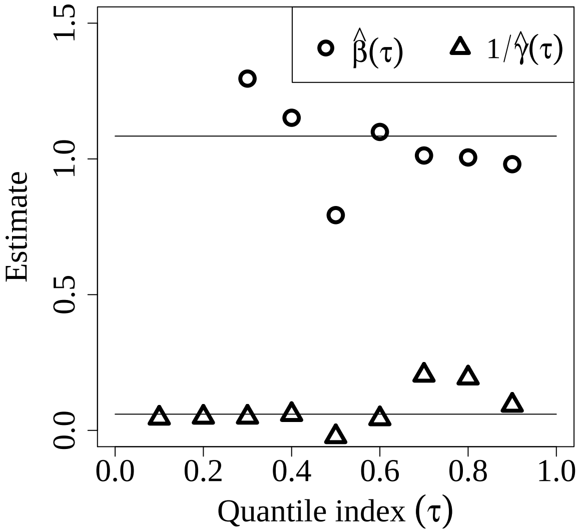
Quantile Euler equation estimates

- Same data, model, instruments as Yogo (2004) Table 2 (but with quantiles); "weak instruments are not a problem" (Yogo, 2004, p. 805)
- Very little smoothing (for estimation)
- β: discount factor ($1 =$ no discount)
- $1/\gamma$: EIS, elasticity of intertemporal substitution

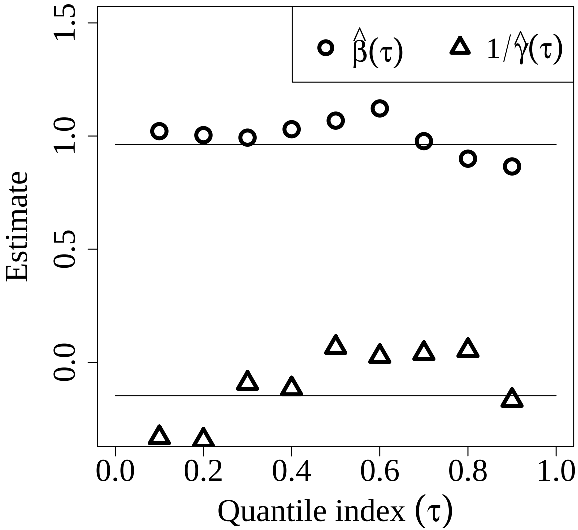
Quantile Euler equation estimates: UK



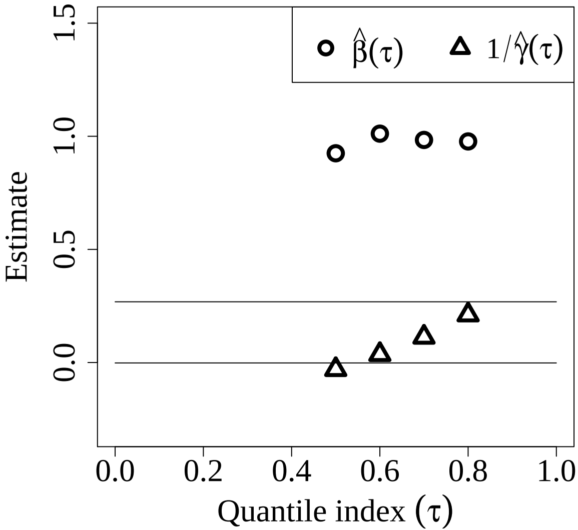
Quantile Euler equation estimates: USA



Quantile Euler equation estimates: Netherlands



Quantile Euler equation estimates: Sweden



Quantile Euler equation estimates

	US		UK		NTH		SWE	
τ	$\hat{\beta}_\tau$	$\hat{\gamma}_\tau$	$\hat{\beta}_\tau$	$\hat{\gamma}_\tau$	$\hat{\beta}_\tau$	$\hat{\gamma}_\tau$	$\hat{\beta}_\tau$	$\hat{\gamma}_\tau$
0.70	1.01	4.9	1.00	2.8	0.98	25.5	0.98	8.9
0.80	1.01	5.2	0.98	3.9	0.90	19.1	0.98	4.7
2SLS	1.08	16.7	1.03	6.0	0.96	−6.8	0.27	−544.4

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Conclusion

- Smoothed IVQR: fast, scalable, robust computation (and better MSE)
- First theoretical results for feasible IVQR with dependent data (and nonlinear model)
- Quantile Euler equations: decouple EIS and risk attitude, robust to fat tails, no error in log-linearization, more reasonable estimates than 2SLS (in our example)
- Many open questions: quantile GMM (in progress)? averaging estimator like in Hansen (2017) or Cheng, Liao, and Shi (2016) (in progress)? semi/nonparametric? uniformity in τ? determination of τ?

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- Thank you!
- (And further questions or comments are welcome)

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