Bayesian and frequentist inequality tests with ordinal data

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Outline

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- SRHS is 1) "useful over the complete adult life cycle" and 2) strongly correlated with more objective measures (mortality, activities of daily living, etc.).
- Interested in "whether inequality in health status. . . increases with age" as well as "across socioeconomic groups."
- "Plausible that health shocks have both permanent and transitory components. . . the former implies that health status will be nonstationary. . . dispersion of health status will grow with age."

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- Goal: "to document the evolution of [SRHS] with age, looking at both cohort means and within-cohort dispersion."
- "Although some health shocks will have only temporary effects, others will leave a permanent residue, so that even if this residue is a small component of the original shock, the resulting health status will be non-stationary. . . . health of members of a cohort will disperse over time."

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Variance

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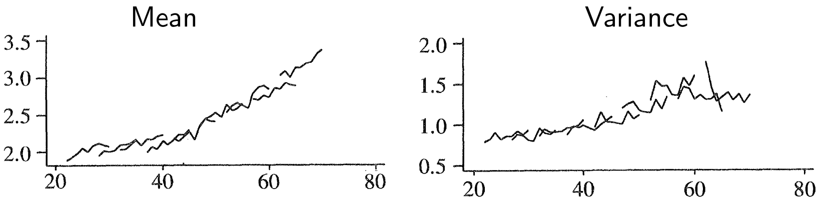

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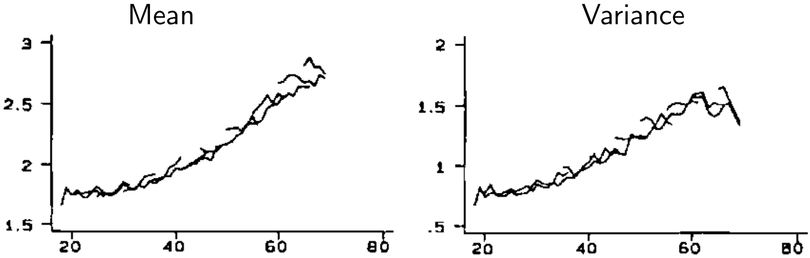

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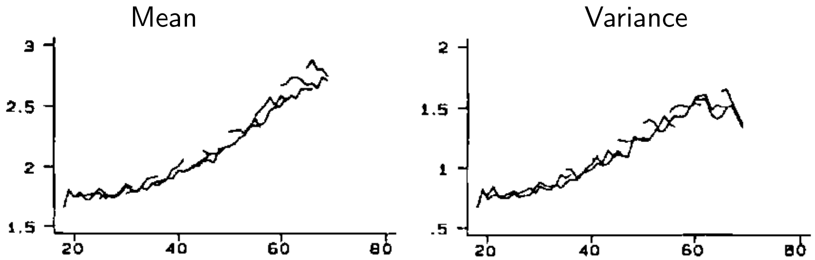

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- Dispersion increases with age?
- But: variance depends on cardinal values; SRHS is ordinal, "values" (1=excellent, . . . , 5=poor) are arbitrary.

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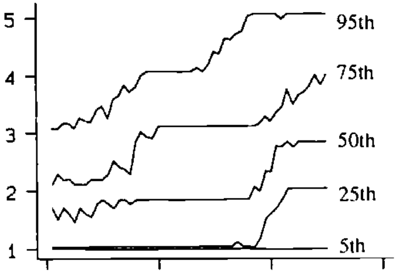

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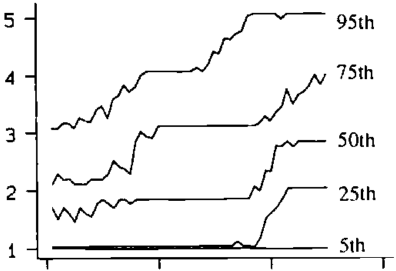
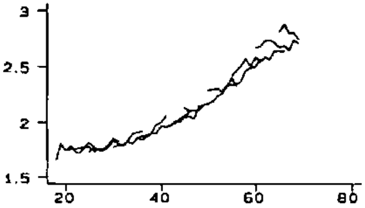


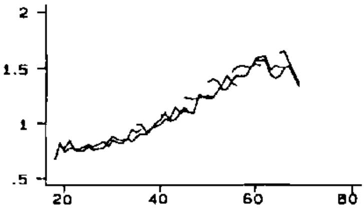
- Doesn't make same "cardinal sin" of variance; but, is dispersion zero if everyone is "excellent" (or "poor")?
- Could the ordinal data trends be explained by a latent health variable whose distribution shifts only in location, not dispersion?

Motivation: SRHS, pure latent location shift

Mean



Variance



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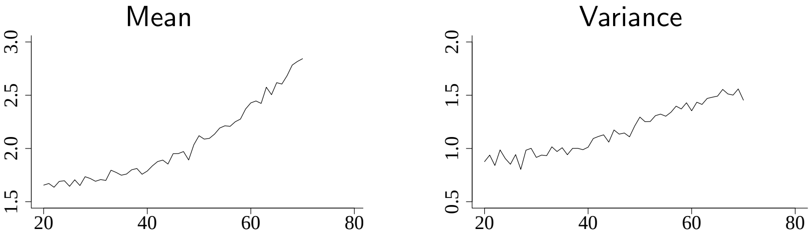

DGP: for ages $a = 20, \ldots, 70$, sample 1000 iid $\mathrm{N}(\mu_a, 1)$ each for increasing μ_a, convert to ordinal using fixed thresholds.

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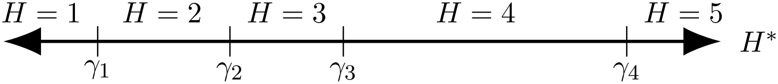
- Assume latent health H^*, SRHS H, fixed thresholds γ_j:



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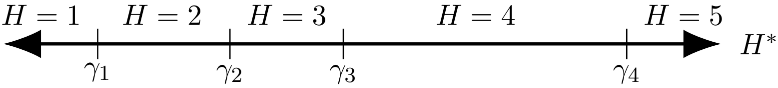

- Pure location shift of $H^* \implies$ SD1 in $H^* \implies$ SD1 in H.
- Proof: picture. $F(j) = F^*(\gamma_j)$, so $F_1^*(\gamma_j) \leq F_2^*(\gamma_j) \iff F_1(j) \leq F_2(j)$

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- Let $H_0\colon H_2 \mathrel{\mathrm{SD}_1} H_1$ (ordinal SD1).
- This is a set of moment inequalities.
- Let $\theta_j \equiv F_2(j) - F_1(j) = \mathbb{E}[\mathbb{1}\{H_2 \leq j\} - \mathbb{1}\{H_1 \leq j\}]$, so $H_0\colon \theta_j \leq 0$ for $j = 1, \ldots, 4$.

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- Can use moment inequality tests.
- Bayesian inference?

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- I was surprised to find Bayesian inference "more conservative" in bivariate normal setup with zero correlation, and even more so with positive correlation, but the opposite with negative correlation.
- Curious to understand why.
- Also computed unbiased frequentist tests using Moreira and Moreira (2013), but dropped this after reading Perlman and Wu (1999).

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- Frequentist "too aggressive": Lindley's (1957) paradox and Berger and Sellke (1987), testing point (or small interval) hypothesis with prior $\mathrm{P}(H_0) = 1/2$; although Casella and Berger (1987b) disagree that $\mathrm{P}(H_0)$ is "objective."

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- Similar/same: Casella and Berger (1987a) one-sided hypothesis (scalar); Berger, Brown, and Wolpert (1994) conditional frequentist.

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- Methods: many possible methods. . . consider frequentist size of a certain Bayesian hypothesis test.
- Limit experiment: ignore influence of prior (for now) (kind of).

Methods: decision-theoretic setup

- Test $H_0 \colon \boldsymbol{\theta} \in \Theta_0$ vs. $H_a \colon \boldsymbol{\theta} \notin \Theta_0$.
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- Thus: PEL if reject is smaller iff $\mathrm{P}(H_0 \mid \mathbf{X}) \leq \alpha$.
- This is the "Bayesian test" we consider.

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- Unbiased frequentist test with size α (e.g., Lehmann and Romano, 2005, Problem 1.10).
- If H_0 true: risk is rejection probability (RP) times $1-\alpha$ loss; RP bounded above by size.
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- Thus: size α gives max risk $\alpha(1-\alpha)$ in both cases.
- Even without unbiasedness, still approximately true (e.g., want size 0.052 instead of $\alpha = 0.05$).

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- $\mathbf{X}$ and $\boldsymbol{\theta}$ in Banach space, possibly infinite-dimensional.
- $\phi(\cdot)$: continuous linear functional.
- Bernstein–von Mises (BvM) theorem: $\phi(\mathbf{X}) - \phi(\boldsymbol{\theta}) \mid \boldsymbol{\theta} \sim F$, $\phi(\boldsymbol{\theta}) - \phi(\mathbf{X}) \mid \mathbf{X} \sim F$
- $F(\cdot)$: continuous CDF, support $\mathbb{R}$, symmetry $F(-x) = 1 - F(x)$.

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- Asymptotics: $\boldsymbol{\theta}$ is local parameter.
- Easier to have BvM with drifting centering value than drifting DGP:

$$\mathbf{X}_n = \sqrt{n}(\hat{\boldsymbol{\mu}} - \boldsymbol{\mu}_{0,n}) = \overbrace{\sqrt{n}(\hat{\boldsymbol{\mu}} - \boldsymbol{\mu})}^{\xrightarrow{d} \mathrm{N}(\mathbf{0}, \underline{\boldsymbol{\Sigma}})} + \overbrace{\sqrt{n}(\boldsymbol{\mu} - \boldsymbol{\mu}_{0,n})}^{\equiv \boldsymbol{\theta}_n \to \boldsymbol{\theta}}$$

$$\xrightarrow{d} \overbrace{\mathbf{X} \sim \mathrm{N}(\boldsymbol{\theta}, \underline{\boldsymbol{\Sigma}})}^{\text{limit experiment}}, \quad \underline{\boldsymbol{\Sigma}} \text{ known or } \hat{\underline{\boldsymbol{\Sigma}}} \xrightarrow{p} \underline{\boldsymbol{\Sigma}}.$$

- Posterior: $\boldsymbol{\theta}_n = \sqrt{n}(\boldsymbol{\mu} - \boldsymbol{\mu}_{0,n})$,

$$\boldsymbol{\theta}_n - \mathbf{X}_n = \sqrt{n}(\boldsymbol{\mu} - \hat{\boldsymbol{\mu}}) \xrightarrow{d} \mathrm{N}(\mathbf{0}, \underline{\boldsymbol{\Sigma}}).$$

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$$\theta \mid X \sim \mathrm{N}\left(\frac{\tau^2 X + m}{\tau^2 + 1}, \frac{\tau^2}{\tau^2 + 1}\right),$$
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- Parametric BvM: Theorem 10.1 in van der Vaart (1998, §10.2) and Theorems 20.1–3 in DasGupta (2008, §20.2).
- Semiparametric BvM: Shen (2002), Bickel and Kleijn (2012), Castillo and Rousseau (2015); Hahn (1997, Thm. G), Kwan (1999, Thm. 2), Kim (2002, Prop. 1), Lancaster (2003, Ex. 2), Schennach (2005, p. 36), Sims (2010, Sec. III.2), Norets (2015, Thm. 1).

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Theorem: part (i), scalar case

- Let $X, \theta \in \mathbb{R}$, $X \sim \mathrm{N}(\theta, 1)$, $\theta \sim \mathrm{N}(X, 1)$, $H_0\colon \theta \leq 0$, $\alpha = 10\%$.

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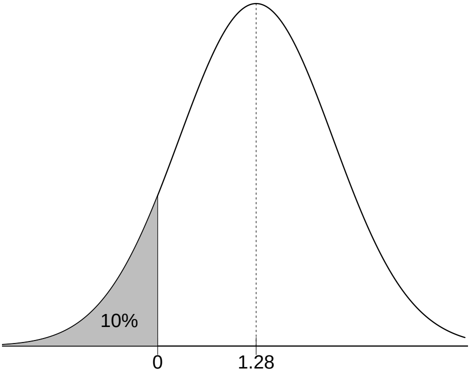
- Posterior of θ given $X = 1.28$:



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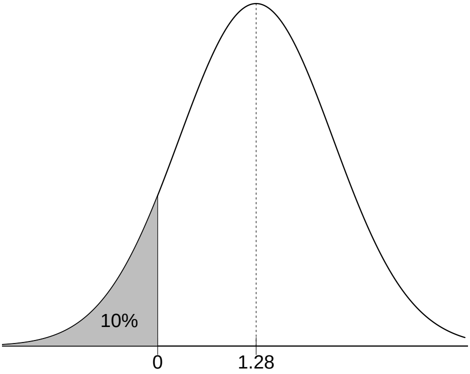

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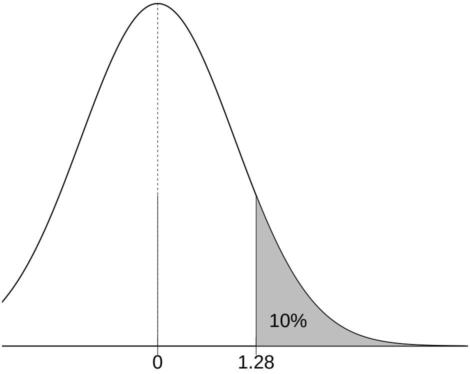
- Sampling distribution of X given $\theta = 0$:



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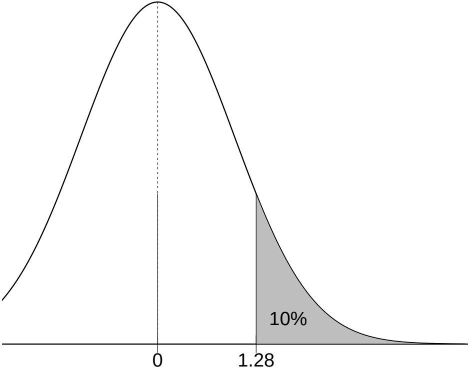
- $\mathrm{P}(H_0 \mid X = 1.28) = \alpha \implies$ reject iff $X \geq 1.28$.
- Size: $\sup_{\theta \leq 0} \mathrm{P}(\text{reject} \mid \theta) = \mathrm{P}(X \geq 1.28 \mid \theta = 0) = \alpha$.



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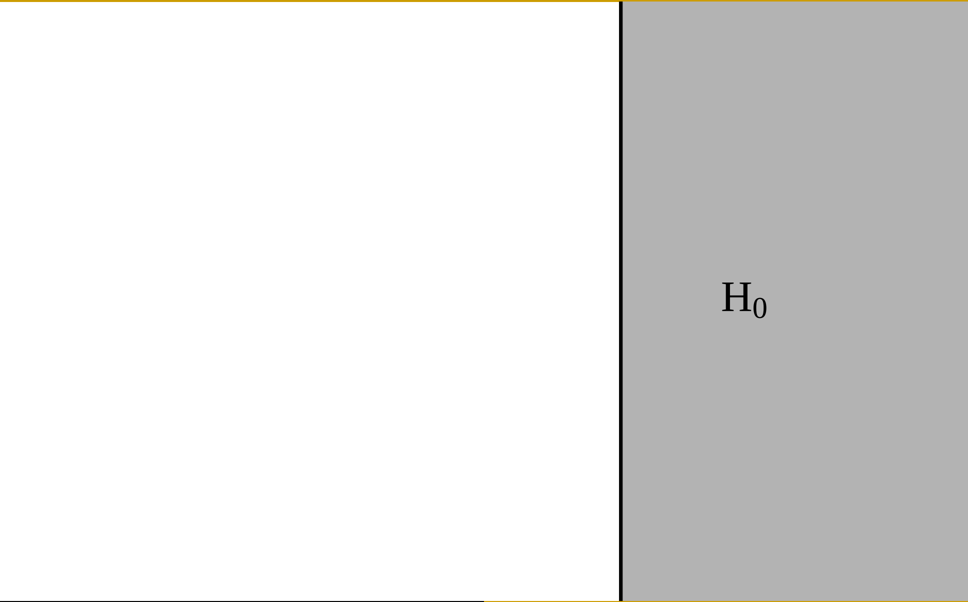
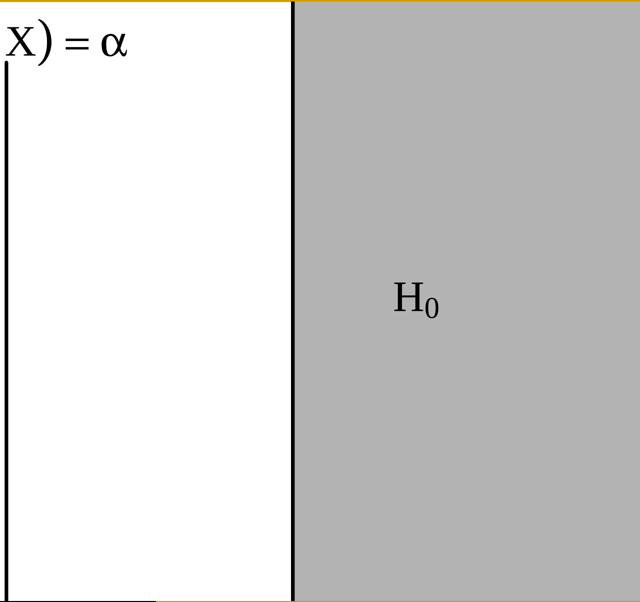
- Ex: $\mathbf{X} \sim \mathrm{N}(\boldsymbol{\theta}, \underline{\mathbf{V}})$, then for constant vector $\mathbf{c}$, $\mathbf{c}'\mathbf{X} \sim \mathrm{N}(\mathbf{c}'\boldsymbol{\theta}, \mathbf{c}'\underline{\mathbf{V}}\mathbf{c})$, scalar Gaussian.
- Ex: if $X(\cdot)$ is Gaussian process, then $X(r)$ is scalar Gaussian. So is $\phi(X(\cdot))$ if ϕ belongs to the dual of the Banach space of $X(\cdot)$.

Theorem: part (ii,iii)



Theorem: part (ii,iii)

$$P(H_0 \mid X) = \alpha$$



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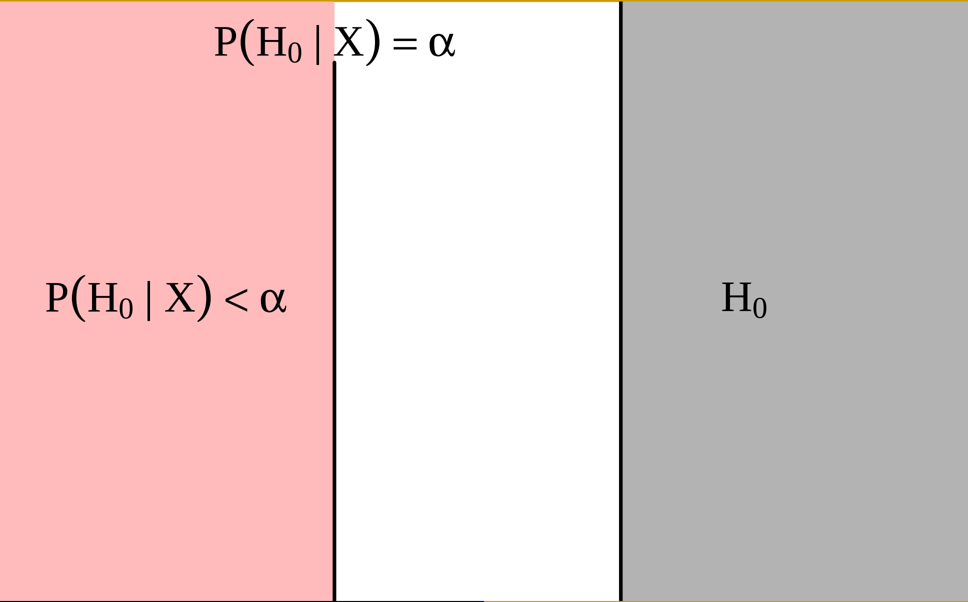

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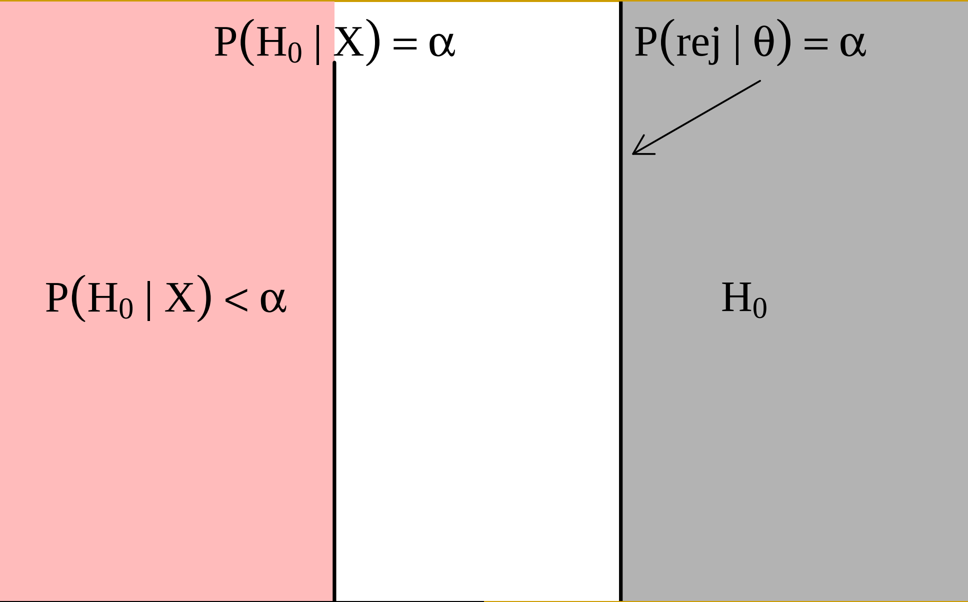

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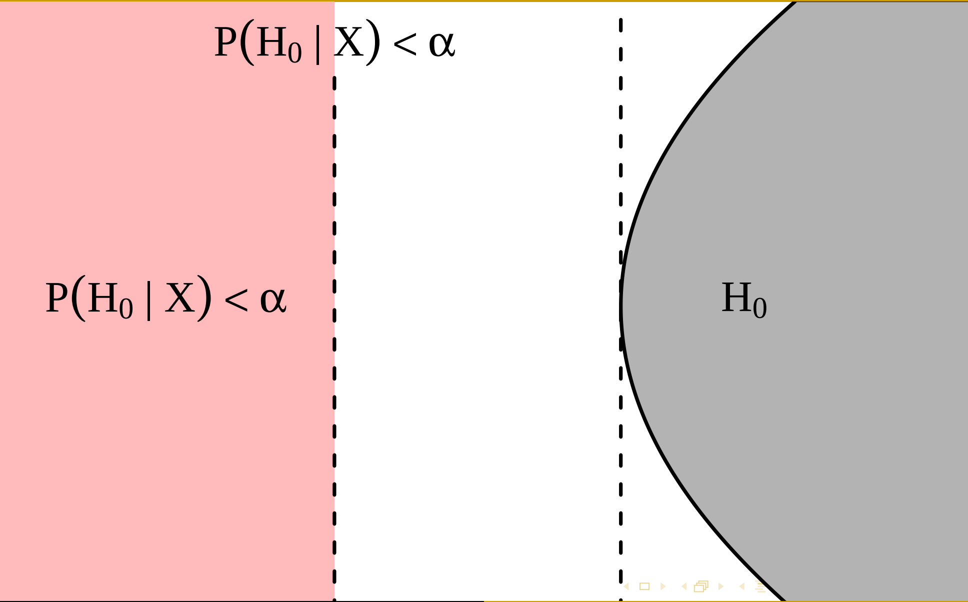

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- What if $d > 2$ dimensions, or infinite?
- $\mathbb{R}^d$: same argument applies if carved away part has positive Lebesgue measure and distribution has support on $\mathbb{R}^d$.
- Infinite: basically, check if there is a finite-dimensional test of a necessary (not sufficient) condition of the infinite-dimensional H_0.
- $A \implies B$ means "reject B" $\implies$ "reject A." So, $\mathrm{P}(\text{rej } B \mid \theta(\cdot)) < \mathrm{P}(\text{rej } A \mid \theta(\cdot))$, and $\mathrm{P}(\text{rej } B \mid \theta(\cdot)) > \alpha \implies \mathrm{P}(\text{rej } A \mid \theta(\cdot)) > \alpha$.
- Ex: $H_0\colon \theta(\cdot) \le 0(\cdot) \implies (\theta(r_1), \theta(r_2)) \le (0, 0)$.

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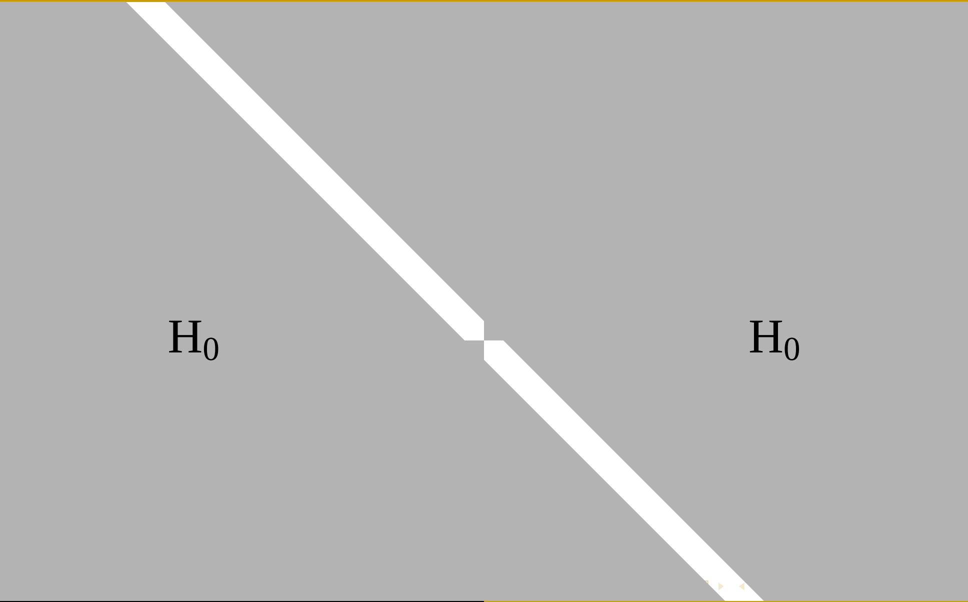
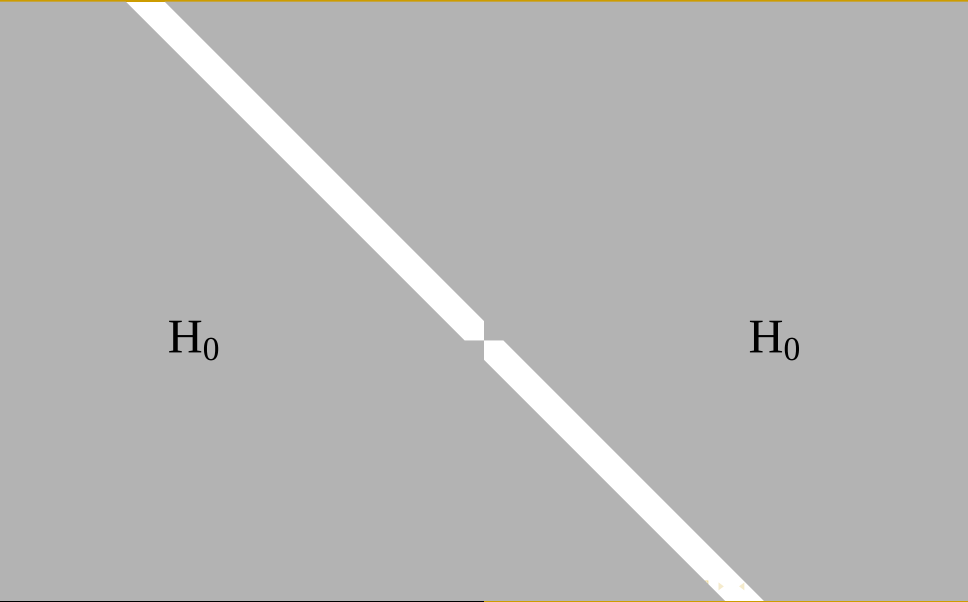
- Anything's possible: size$\{>, =, <\}\alpha$.
- Further: may depend on distribution, not just shape of Θ_0.
- Examples follow: bivariate normal distribution, unit variances, correlation ρ.

Thm(iv): example of size depending on ρ



Thm(iv): size is 0% if $\rho = 1$ ($\mathrm{P}(H_0 \mid \mathbf{X}) = 1$, $\forall \mathbf{X}$)



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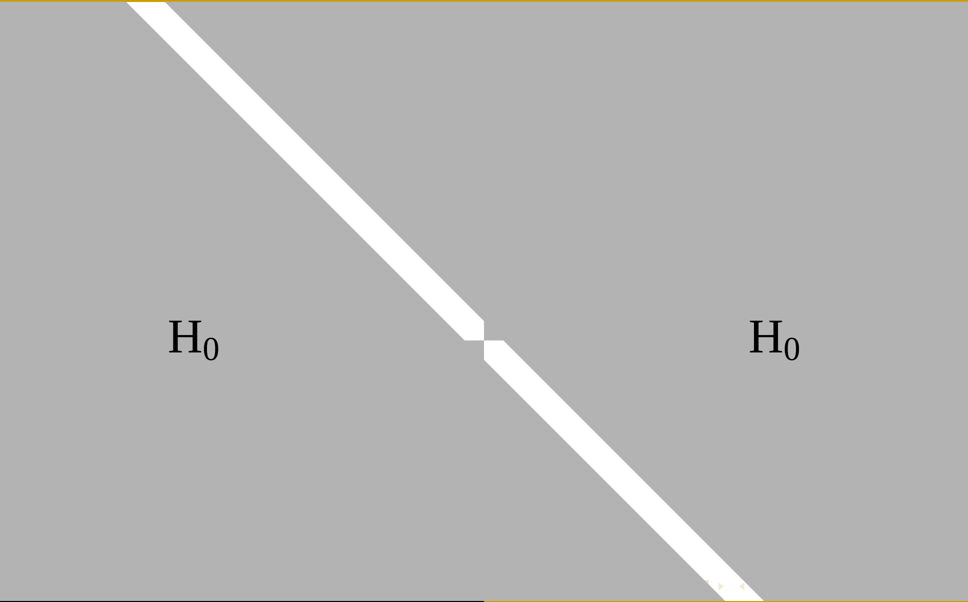
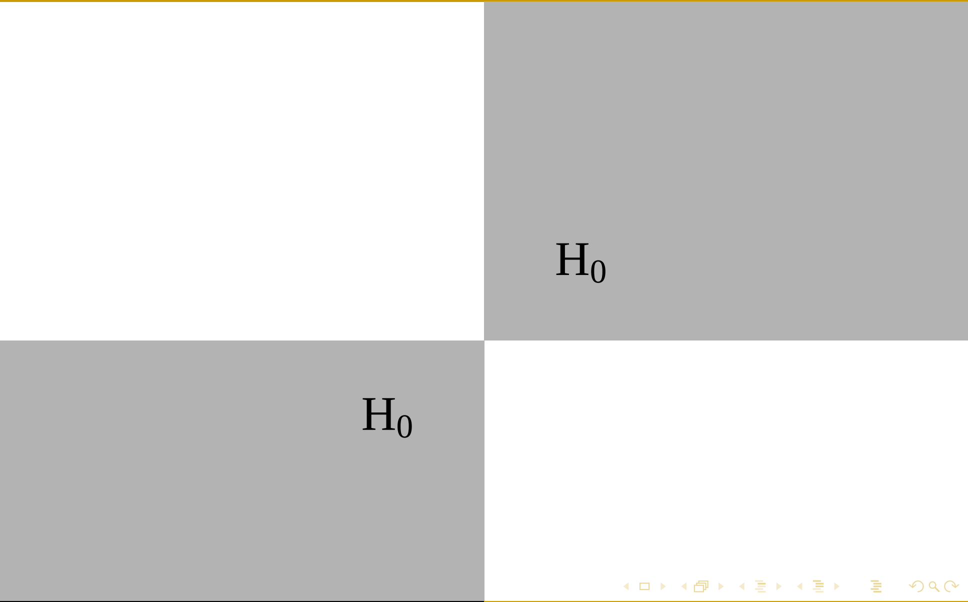


Thm(iv): example of $H_0\colon \theta_1\theta_2 \geq 0$



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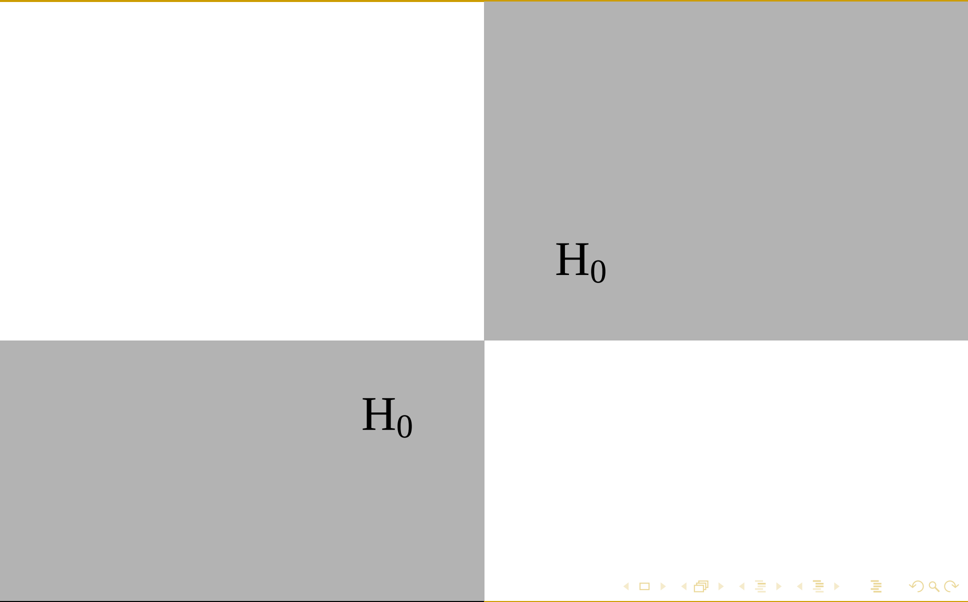
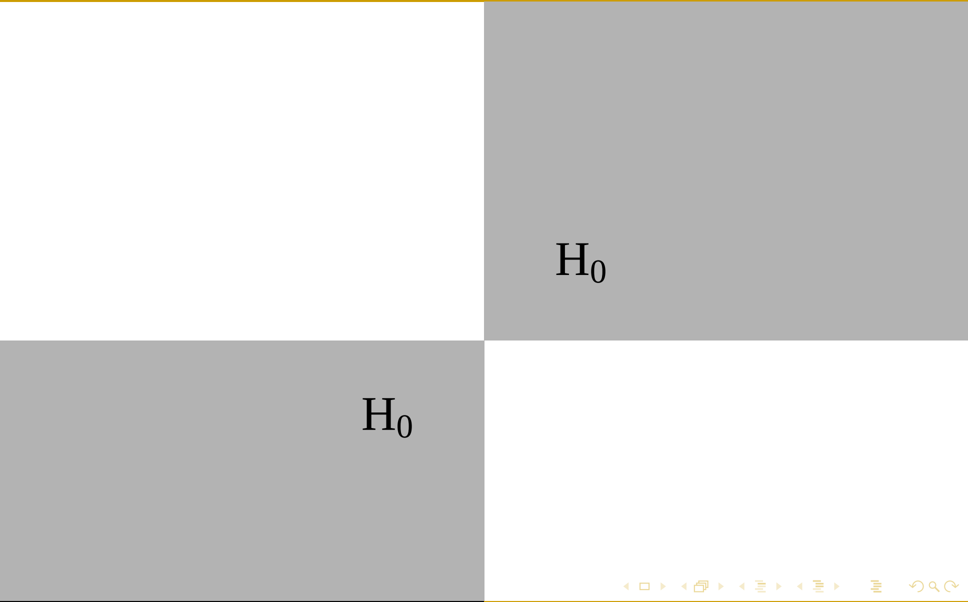


Thm(iv): size is α if $\rho = 1$ (let $\theta_1 \to \infty$, $\theta_2 = 0$)



Theorem: discussion

- Part (iii) is partly due to small prior $\mathrm{P}(H_0)$.
- Flat prior on $\boldsymbol{\theta}$ means $\mathrm{P}(H_0)$ changes with H_0.
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- Flat prior on $\boldsymbol{\theta}$ means $\mathrm{P}(H_0)$ changes with H_0.
- Would be interesting to examine $\mathrm{P}(H_0) = 1/2$ like J. Berger et al.
- But: not the only factor; shape still important.
- Ex: $H_0 : \theta_1\theta_2 \geq 0$ vs. $H_0 : \theta_1 \leq 0$.

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Theorem: discussion

- Theorem applies to $\Theta_0 = \{\boldsymbol{\theta} : g(\boldsymbol{\theta}) \leq g_0\}$ as special case, but $g(\cdot)$ not restricted to directional differentiability, etc.
- Another special case: if multiple linear inequalities, then size strictly above α. (If single, then size α.)
- It matters greatly whether $H_0\colon \boldsymbol{\theta} \in \Theta_0$ or $H_0\colon \boldsymbol{\theta} \notin \Theta_0$: if part (iii) applies to $H_0\colon \boldsymbol{\theta} \in \Theta_0$, then part (iv) applies to $H_0\colon \boldsymbol{\theta} \notin \Theta_0$.
- Ex: stochastic dominance.

Outline

First-order stochastic dominance (SD1)

- One-sample SD1 (see paper for two-sample):
$F_X(\cdot) \leq F_0(\cdot) \iff \sqrt{n}(F_X(\cdot) - F_0(\cdot)) \leq 0(\cdot)$.

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- Like before, $X_n(\cdot) = \sqrt{n}(\hat{F}_X(\cdot) - F_{0,n}(\cdot))$, $\theta_n(\cdot) = \sqrt{n}(F_X(\cdot) - F_{0,n}(\cdot))$.
- Limit experiment: $H_0\colon \theta(\cdot) \le 0(\cdot)$, $X(\cdot) - \theta(\cdot) \mid \theta(\cdot) \sim B(F_X(\cdot))$.
- Theorem part (iii) applies here, when H_0 is SD1. (Bayesian test of necessary condition $(\theta(r_1), \theta(r_2)) \le (0, 0)$ has size above α already.) But, (iv) applies if H_0 is non-SD1.

SD1: analytic results

- Prop. 2: $\mathrm{P}(\mathrm{SD}_1 \mid X(\cdot) = 0(\cdot)) = 0$. (Similar to p-value comparisons in Kline (2011).)
- Prop. 3: Bayesian test's type I error rate is 100% when $\theta(\cdot) = 0(\cdot)$.
- Cor. 4: rejection probability is zero for non-SD1 when $\theta(\cdot) = 0(\cdot)$.

SD1: simulations (fixed dataset)

Posterior probs, $X_i = i/(n+1)$ vs. Unif(0,1) (or vs. $Y_i = i/n$).

		Comparison distribution	
H_0	n	$\mathrm{Unif}(0,1)$	Y
SD1	10	0.103	0.097
SD1	40	0.028	0.025
SD1	100	0.009	0.010
SD1	∞	0.000	0.000

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non-SD1	10	0.897	0.903
non-SD1	40	0.972	0.975
non-SD1	100	0.991	0.990
non-SD1	∞	1.000	1.000

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Curvature: background

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- O'Donnell and Coelli (2005): Bayesian approach appealing for testing curvature due to relative simplicity.
- Here: test concavity of cost function wrt input prices.
- Translog functional form (Christensen, Jorgenson, and Lau, 1973): parametric, but flexible enough to allow violations.

Curvature: translog model (Christensen et al., 1973)

- Output y; input prices $\mathbf{w} = (w_1, w_2, w_3)$; total cost $C(y, \mathbf{w})$.

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- $C(y, \mathbf{w})$ concave in $\mathbf{w}$ (e.g., Kreps, 1990, §7.3).
- $\implies$ Hessian matrix (of C wrt $\mathbf{w}$) negative semidefinite (NSD).
- Here: test "local" NSD at $(1, 1, 1, 1)$; necessary (not sufficient) for global NSD. (Check signs of principal minors. . . .)

Curvature: simulation setup

- Parameters: impose homogeneity of degree one in prices (also imposed in estimation).
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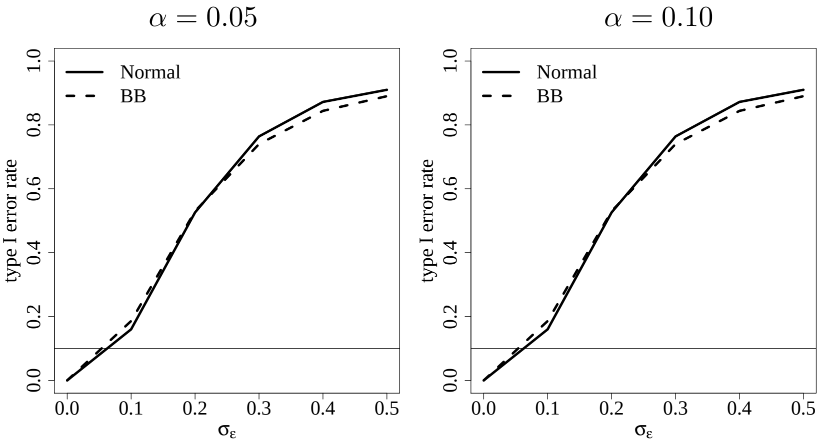
- Other: $n = 100$ observations, 500 simulation replications, 200 posterior draws.

Curvature: simulated type I error rates



Outline

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- "For the case of ordinal health measures, which are arguably more widely employed, dominance results are generally less applicable, there are fewer inequality indices and statistical inference is less well developed."
- "The breakthrough in analyzing inequality with such data [like SRHS] came from Allison and Foster (2004)."

Ordinal dispersion: median-preserving spread (MPS)

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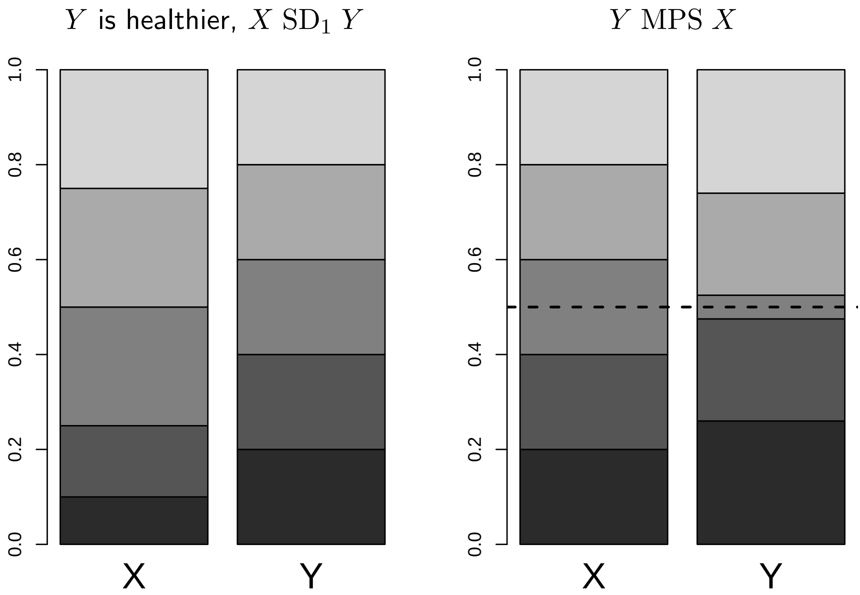
- Like SD1, may be written as set of moment inequalities.
- m: shared median; i.e., $F_1(m-1) < 1/2 \leq F_1(m)$, same for F_2.
- For $j < m$, $F_1(j) \leq F_2(j)$; for $j \geq m$, $F_1(j) \geq F_2(j)$.
- Let $\theta_j \equiv F_1(j) - F_2(j)$ again.
- MPS is $\theta_j \leq 0$ for $j < m$ and $\theta_j \geq 0$ for $j \geq m$.



Y is healthier, $X \text{ SD}_1 Y$
1.0
0.8
0.6
0.4
0.2
0.0
X
Y
$Y \text{ MPS } X$
1.0
0.8
0.6
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- "Y MPS X" means Y is a MPS of X.
- Frequentist: Andrews and Barwick (2012) refined moment selection (RMS).
- Bayesian: Dirichlet–multinomial model, uninformative prior.

Empirical: SD1 and MPS over age

X: 20–24 years old; Y: age range in table.

p-values and posterior probabilities; **bold** if below 5%.

	$H_0: Y \text{ SD}_1 X$		$H_0: X \text{ SD}_1 Y$	
Y	RMS	Bayes	RMS	Bayes
$[25, 29]$	8.0%	**0.8%**	63.6%	13.1%
$[30, 34]$	100%	42.3%	**3.7%**	**0.1%**
$[35, 39]$	100%	74.7%	**0.1%**	**0.0%**
$[40, 44]$	100%	86.0%	**0.0%**	**0.0%**

Empirical: SD1 and MPS over age

X: 20–24 years old; Y: 25–29 years old.

H_0 in table header. (MPS: same median, "very good.")

	$Y\ \mathrm{SD}_1\ X$	$X\ \mathrm{SD}_1\ Y$	Y MPS X	X MPS Y
RMS	8.0%	63.6%	37.2%	9.3%
Bayes	**0.8%**	13.1%	7.7%	**0.0%**

Empirical: SD1 and MPS over generation (at same age)

Ages 65–69. X born 1937–41, Y born 1932–36.

B=“black”; W=“white”; M=“male”; F=“female”

	$H_0: Y$ MPS X		$H_0: Y$ SD$_1$ X		$H_0: X$ SD$_1$ Y	
Sample	RMS	Bayes	RMS	Bayes	RMS	Bayes
B	100%	25.8%	39.7%	**3.2%**	100%	17.9%
BM	76.4%	11.7%	12.4%	**0.1%**	100%	39.4%
BF	100%	10.3%	100%	25.4%	61.2%	**4.3%**

Empirical: SD1 and MPS over generation (at same age)

Ages 65–69. X born 1947–51, Y born 1942–46.

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	$H_0: Y$ MPS X		$H_0: Y$ SD$_1$ X		$H_0: X$ SD$_1$ Y	
Sample	RMS	Bayes	RMS	Bayes	RMS	Bayes
M	100%	26.9%	51.4%	9.3%	58.9%	6.8%
BM	100%	38.3%	**4.6%**	**1.2%**	16.4%	**2.0%**
WM	100%	12.4%	100%	15.8%	65.3%	10.3%

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- Middle ground (kind of): take latent health seriously, but only impose shape restrictions instead of fully parametric; how much can shape restrictions buy?
- Of course, all assuming fixed thresholds, or for SD1 allow parallel shift ("index shift"); some evidence of index shift (e.g., Hernández-Quevedo, Jones, and Rice, 2005), some of stability (PSID asking older respondents to rate their younger selves' SRHS). Maybe some non-health ordinal settings where thresholds are more stable?

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- Can never learn about tails or tail-sensitive measures (like variance) from ordinal data; what can we learn from CDFs evaluated at $\gamma_1, \ldots, \gamma_4$?

Ordinal dispersion: location-scale model

- Example: assume latent location-scale model:
  $F_1^*(x) = F^*((x - \mu_1)/\sigma_1)$, $F_2^*(x) = F^*((x - \mu_2)/\sigma_2)$.

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- Implies single crossing of CDFs (or none if $\sigma_1 = \sigma_2$): crossing is where $F_1^*(r) = F_2^*(r)$, so $(r - \mu_1)/\sigma_1 = (r - \mu_2)/\sigma_2, \ldots,$ $r = (\sigma_2\mu_1 - \sigma_1\mu_2)/(\sigma_2 - \sigma_1)$.

Ordinal dispersion: location-scale model

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- Implies single crossing of CDFs (or none if $\sigma_1 = \sigma_2$): crossing is where $F_1^*(r) = F_2^*(r)$, so $(r - \mu_1)/\sigma_1 = (r - \mu_2)/\sigma_2, \ldots,$ $r = (\sigma_2\mu_1 - \sigma_1\mu_2)/(\sigma_2 - \sigma_1)$.
- For any $s > r$, $(s - \mu)/\sigma = (r - \mu)/\sigma + (s - r)/\sigma$. If $\sigma_1 < \sigma_2$, then $(s - r)/\sigma_1 > (s - r)/\sigma_2$, so $(s - \mu_1)/\sigma_1 > (s - \mu_2)/\sigma_2$ since $(r - \mu_1)/\sigma_1 = (r - \mu_2)/\sigma_2$ by definition of r. Thus, $F_1^*(s) \geq F_2^*(s)$ for any $s > r$.

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- Similarly, $F_1^*(s) \leq F_2^*(s)$ for any $s < r$, again letting $\sigma_1 < \sigma_2$.

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- Similarly, $F_1^*(s) \leq F_2^*(s)$ for any $s < r$, again letting $\sigma_1 < \sigma_2$.
- So, if the ordinal CDFs cross so that $F_1(j) < F_2(j)$ and $F_1(k) > F_2(k)$ for ordinal categories $j < k$, then $F_1^*(\gamma_j) < F_2^*(\gamma_j)$ and $F_1^*(\gamma_k) > F_2^*(\gamma_k)$, implying $\gamma_j < r < \gamma_k$ and $\sigma_1 < \sigma_2$.

Ordinal dispersion: location-scale model (continued)

- But: in some cases like SRHS vs. age, either $r < \gamma_1$ or $r > \gamma_4$, so there is no ordinal CDF crossing even if there's a latent CDF crossing.

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Ordinal dispersion: location-scale model (continued)

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- And: multiple crossings would suggest misspecification.
- But: just a way to motivate the use of a single-crossing property to infer "smaller latent dispersion."
- And: MPS is special case, where $\gamma_{m-1} < r < \gamma_m$.

Outline

Conclusion

- Summary (Bayes/freq): if null hypothesis is smaller than half-space, then Bayesian test has (asymptotic) size above α; if not, then can depend on sampling distribution, too.
- Summary (ordinal): can characterize some relationships between ordinal distributions as set of moment inequalities; additional assumptions imply latent distribution relationships.

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- Thank you!
- (And further questions or comments are welcome)

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