1 Introduction

1.1 Background. Variable-displacement, swash-plate type, axial-piston pumps are widely used in industrial and mobile applications for supplying power to hydraulic circuitry. In typical off-highway applications, these pumps are used in excavators and wheel loaders to convert rotational power from an engine to hydraulic power that is used in an implement circuit for raising and lowering a bucket. In other applications, axial-piston pumps may be used for providing hydraulic power to a hydrostatic transmission that then propels a vehicle or operates an industrial lathe. Many other applications within the automotive, agricultural, and industrial sectors abound. A frequent control-configuration for these machines is to maintain a constant discharge pressure while providing a variable flow-rate that changes depending upon the immediate demand of the application. In this arrangement, the variable-displacement axial-piston pump is used to provide a power source analogous to household electrical-power with a constant voltage and variable current.

Axial-piston pumps have a number of attributes that make them attractive for high-power applications. The most advantageous feature of an axial-piston pump over, say, an electrical device is that it exhibits a very high effort-to-inertia ratio which tends to create a stiff dynamical-response. A companion attribute of this feature is that hydraulic machinery converts and transmits large amounts of power using small devices—an attribute often referred to as “power density.” Disadvantages of using axial-piston pumps for converting power include low efficiency compared to the process for generating electricity and the messiness of using hydraulic fluid. Another occasional drawback for using hydraulic pump technology is that historical pump bandwidth-frequencies have been limited to around 25 Hz in practice. While this has proved sufficient for many applications, modern high-speed applications are occasionally demanding a bandwidth frequency of 50–100 Hz. The aim of this present research is to investigate the physical parameters that are responsible for limiting the bandwidth frequency of a pressure controlled, axial-piston pump.

1.2 Literature Review. Much of the early effort toward understanding the performance of axial-piston pumps has been focused on the steady-state pump characteristics and only in the last 30 years have the dynamical characteristics been made an object of research. Recently, analysis has been done to understand and improve the dynamic behavior of axial-piston pumps when controlled by a hydraulic input signal. For instance, Baz [1] has recommended the use of a high natural-frequency stroking mechanism in order to minimize the steady-state error, peak pressure-overshoot, and the settling time. Baz summarizes previous work in his article and suggests that the past emphasis has been placed on optimizing the frequency response of the pump at specific operating conditions rather than synthesizing the overall pump behavior to obtain specific performance under various load disturbances. Akers and Lin [2–4] have investigated applications of optimal control theory for a pump utilizing a single-stage and a two-stage servo valve. In his work, he developed an optimal control law for such a system and found that the response frequency obtained by using a two-stage valve was slightly faster when compared to a single-stage valve and that the peak pressures were reduced by a factor of 2. Zeiger and Akers [5] also developed a seminal piece of work for describing the basic dynamics for the swash-plate of an axial-piston machine. Schoenau et al. [6] developed the first mathematical model aimed at predicting both the steady-state characteristics and the dynamic-response characteristics of the pump.
In his work, the model was subjected to a simulated pressure control signal and the output of the swash-plate angle was compared to an experimentally generated displacement time-trace. A good correlation between experiments and theory was reported. Manring and Johnson [7] developed several simple equations to govern the design of actuators and the control gain of a variable-displacement pump. In this work, the authors recorded the effect of varying the actuator volume, discharge hose volume, and the control gain in terms of settling time, rise time, and maximum percent overshoot. This research reported that the stability of the pump depends on the competing effects between destabilizing forces of the discharge pressure and the restoring forces of the control pressure but the impact on bandwidth frequency was not investigated. Dobchuk et al. [8] presented a very comprehensive literature review detailing chronological progress that would be most helpful in developing a complete and accurate mathematical model of an axial-piston pump. In his work, the authors identified the need to solve the high frequency response problem numerically as he considered the fourth-order linear model developed by Akers to be insufficient. Recently, Leslie et al. [8] has described the use of dynamic neural networks with dynamic neural units in simulating a variable-displacement pump necessary to capture or represent some of the dynamics of a plant, which a static (feed forward) neuron cannot do. Manring et al. [10–12] has also published more recent work in which various machine design-features have been examined for controlling the swash-plate of the pump; but, again, the bandwidth frequency of the pump was not examined.

While all of this work has been useful for considering the complex behavior of axial-piston pumps, none of it has been aimed at investigating the fundamental and physical limitations that have been observed when trying to increase the bandwidth frequency of the pump beyond, say, 25 Hz. This current research seeks to augment the literature by investigating this topic from a theoretical point of view using nondimensional analysis for studying the sensitivity of the pump beyond, say, 25 Hz. Though much research has been done over the past 30 years to understand the dynamical behavior of these machines, the essential design characteristics that determine the bandwidth frequency of the pump remain elusive. In part, this is due to the fact that the machine is complex and when coupled with a hydraulic control valve that is disturbed by steady and transient fluid-momentum effects this dynamical property becomes difficult to assess. In order to achieve the objectives of this research, this paper presents the most comprehensive pump-and-valve model of a pressure controlled, axial-piston machine available in the literature to date. The pump model includes the effects of the discrete pumping-elements acting on the swash plate, while the valve model includes both steady and transient fluid-momentum forces. To identify the dominant features of the model, nondimensional analysis is employed and the complexity of the model is subsequently reduced by eliminating negligible terms. Furthermore, a closed-form expression for the bandwidth frequency is employed and perturbation analysis is used to identify the dominant set of parameters that impact the bandwidth frequency of the pump. In conclusion, it is shown that, by far, the greatest impact on the bandwidth frequency may be achieved by reducing the swept volume of the control actuator and by increasing the flow capacity of the control valve.

1.4 Pump Description. Figure 1 shows a cross-sectional view of the pressure controlled, axial-piston, swash-plate type, variable-displacement pump. The pump itself is constructed using pistons that are nested in a circular array within a cylinder block. The cylinder block is attached to an input shaft which is then driven by a turning power-source in order to rotate the pistons and cylinder block relative to a fixed valve plate, which separates the intake port from the discharge port of the pump. A sectional view of the valve plate, Section A-A, is shown in Fig. 3. The ball-and-socket joint of each piston is attached to a slipper (not labeled in Fig. 1) which then slides against the flat surface of the swash plate. The fluid pressure within the cylinder block is used primarily to hold the slippers against the swash plate; however, a retaining mechanism is also employed to ensure contact between these parts as shown by various mechanical elements in Fig. 1. As the

![Fig. 1 A schematic of the pressure controlled, axial-piston, swash-plate type, variable-displacement pump (Section A-A is shown in Fig. 3)](image-url)
The swash-plate angle of the pump is shown in Fig. 1 by the symbol $\alpha$. As one may surmise from the previous description, this angle is necessary for achieving the reciprocal motion of the pistons, and its magnitude is used to regulate the amount of fluid that is displaced by the pump. When the swash-plate angle is small, the fluid displaced by the pump is small. When the swash-plate angle is large, the fluid displaced by the pump is large. Therefore, the control of the swash-plate angle is the key objective for controlling the pump displacement and this is achieved by energizing two actuators that exert a rotational moment on the swash-plate about its fixed rotational-axis. These two actuators are identified in Fig. 1 as the bias actuator and the control actuator. As shown in Fig. 1, the bias actuator is energized by a preloaded spring and by the pump discharge pressure that continually forces the swash-plate into stroke. The control actuator is energized by a regulated control-pressure that tends to force the swash-plate into a destroked position. The moments exerted on the swash-plate by the two actuators are combined with a natural pumping-moment that is generated by the pistons and slippers as they rotate within the machine. In order to achieve an equilibrium position for the swash-plate under steady-state conditions, these moments must balance each other.

The left-hand-side of Fig. 1 shows a three-way spool valve that is used to provide the control pressure for the control actuator within the pump. The objective of this valve is to command the control pressure in such a way as to keep the discharge pressure of the pump at a constant and predetermined value. As such, the input signal to the valve is the pump discharge pressure shown in Fig. 1 as it acts against the bottom surface of the spool valve. This pressure forces the spool valve in the positive $\xi$-direction and works against a spring force shown in Fig. 1, which acts downward on the top of the valve. As the discharge pressure rises above its desired value, the spool valve moves in the positive $\xi$-direction thus opening the load port on the three-way valve to the discharge pressure itself, which then causes fluid to flow into the control actuator for destroking the pump. The destroking of the pump then reduces the discharge pressure of the machine causing the valve to return to its neutral position. If the discharge pressure drops below its desired value, the top spring forces the valve to move in the negative $\xi$-direction thus opening the load port on the three-way valve to the return line, which then causes fluid to flow out of the control actuator and the pump stroke then increases. The increasing of the pump stroke then causes the discharge pressure of the machine to rise and the valve returns to its neutral position. Thus a continual metering of fluid in and out of the control actuator is used to adjust the swash-plate angle of the pump, which then alters the discharge flow-rate of the pump, which then seeks to hold the discharge pressure of the pump constant. The following sections of this paper will be used to study the dynamical characteristics of this control objective.

### 1.5 Analysis

#### 1.5.1 General

In this section, the governing equations for the dynamical system will be presented and derived. The objective of the control is to regulate the output pressure of the pump by controlling the pump flow-rate into the discharge line. The analysis of this section will follow the order of reverse causality, beginning with the model of the discharge pressure, moving to the pump model which adjusts the discharge pressure, moving to the pump actuator model which adjusts the pump flow, and concluding with the spool valve model which adjusts the fluid pressure in

\[ ]
the control actuator. Some of the elements of the following analysis can be found in published textbooks on related topics [13–15]. These elements will be identified throughout the paper.

1.5.2 Discharge Pressure. Figure 2 shows a schematic of the pump discharge line. The discharge pressure of the pump is modeled using a common control-volume approach, which assumes that the fluid pressure within the discharge chamber is homogeneous throughout, and that the fluid inertia and viscous effects within this chamber are negligible compared to the hydrostatic pressure effects of the fluid. Based upon these assumptions, the conservation of mass and the definition of the fluid bulk modulus-of-elasticity may be used to model the discharge pressure with the following equation

\[ \frac{V_b}{\beta} \frac{dP_d}{dt} = Q_o - Q_d - KP_d \]  

(1)

where \( V_b \) is the fluid volume in the discharge chamber, \( \beta \) is the fluid bulk-modulus, \( P_d \) is the discharge pressure of the pump, \( t \) is time, \( Q_o \) is the volumetric flow-rate from the pump into the discharge line, \( Q_d \) is the volumetric flow-rate being drawn from the discharge chamber by the downstream load, and \( K \) is the low Reynolds-number coefficient-of-leakage used to describe the volumetric flow-rate that bleeds away from the power transmission-path through clearances in the machinery. In this study, \( Q_d \) is assumed to be a constant. From the steady-state operating conditions of the pump, it may be shown that

\[ Q_d = Q_{do} - KP_{do} \]  

(2)

where \( Q_{do} \) is the steady-state volumetric flow-rate from the pump, and \( P_{do} \) is the steady-state discharge pressure of the pump. The pump flow \( Q_p \) for a swash-plate type axial-piston pump varies with the swash-plate angle \( x \) and has been derived in previous literature [14,15]. This result is presented here as follows

\[ Q_p = \frac{NA_p r \alpha \tan(x)}{\pi} \]  

(3)

where \( N \) is the number of pistons in the machine, \( A_p \) is the cross-sectional area of a single piston within the pump, \( r \) is the piston pitch-radius of the cylinder block, \( \alpha \) is the angular velocity of the input shaft, and \( x \) is the swash-plate angle. For a pump with a fixed input-speed, the volumetric flow-rate generated by the pump is often expressed as

\[ Q_p = G_p x \quad \text{where} \quad G_p = \frac{NA_p r \alpha}{\pi} \]  

(4)

In Eq. (4) the symbol \( G_p \) is called the pump flow-gain and the flow result has been linearized for small values of \( x \). Substituting Eqs. (2) and (4) into Eq. (1) produces the following result for modeling the discharge pressure of the pump

\[ \frac{V_b}{\beta} \frac{dP_d}{dt} + K (P_d - P_{do}) = G_p (x - x_o) \]  

(5)

In this equation \( x_o \) is the steady-state swash-plate angle of the pump.

1.5.3 Swash-Plate Angle. As shown by Eq. (5), the discharge flow of the pump is changed by altering the swash-plate angle of the pump. Figure 3 shows a partial free-body-diagram of the swash-plate with forces being exerted on the swash-plate by the control actuator, the bias actuator, and the \( n \)th piston-slipper assembly. Since the swash-plate pivots about the fixed \( y \)-axis, moments exerted on the swash-plate may be summed about point \( O \) and the equation-of-motion for the swash-plate may be written as

\[ I \frac{d^2x}{dt^2} = -C \frac{dx}{dt} + F_b L_b - F_c L_c - \sum_{n=1}^{N} R_n I_n \]  

(6)

where \( I \) is the mass moment-of-inertia for the swash-plate about the pivot axis, \( C \) is the effective viscous-drag coefficient for the swash plate, \( F_b \) is the force exerted on the swash-plate by the bias actuator, \( F_c \) is the force exerted on the swash-plate by the control actuator, \( R_n \) is the reaction force between the \( n \)th slipper and the swash plate, and the dimensions \( L_b, L_c, \) and \( I_n \) are shown in Fig. 3.

As shown in Eq. (6), the forces exerted on the swash-plate by the actuators and pistons must be modeled to complete the equation-of-motion for the swash plate. Figure 4 shows a partial free-body-diagram for the actuators and pistons. From this figure, the equation-of-motion for the bias actuator may be written as

\[ M_b \frac{d^2x_b}{dt^2} = F_b \cos(x) - P_d A_b - F_o - k x_b \]  

(7)

where \( M_b \) is the mass of the bias actuator, \( A_b \) is the cross-sectional area of the bias actuator, which is continually exposed to the discharge pressure (see Fig. 1), \( F_o \) is the compressed force of the bias-spring when the swash-plate angle is zero, and \( k \) is the bias-spring rate. From the geometry shown in Fig. 3, the position of the bias actuator may be written as

\[ x_b = -L \tan(x) \]  

(8)

For small swash-plate angles, Eq. (8) may be substituted into Eq. (7) and terms may be rearranged to produce the following result for the bias force acting on the swash plate

\[ F_b = -M_b L \frac{d^2x}{dt^2} + P_d A_b + F_o - k L x \]  

(9)

Similarly, Fig. 4 may be used to write the equation-of-motion for the control actuator as follows

\[ M_c \frac{d^2x_c}{dt^2} = F_c \cos(x) - P_c A_c \]  

(10)

where \( M_c \) is the mass of the control actuator and \( A_c \) is the cross-sectional area of the control actuator which is continually exposed to the control pressure. From the geometry shown in Fig. 3, the position of the control actuator may be written as

\[ x_c = L \tan(x) \]  

(11)
For small swash-plate angles, Eq. (11) may be substituted into Eq. (10) and terms may be rearranged to produce the following result for the control force acting on the swash plate

\[ F_c = M_c L \frac{d^2x}{dt^2} + P_c A_c \]  

(12)

Again, using Fig. 4, an equation-of-motion for the nth piston as it moves in the x-direction may be written as

\[ M_p \frac{d^2x_n}{dt^2} = R_n \cos(\theta_n) - P_n A_p \]  

(13)

where \( M_p \) is the mass of a single piston and \( A_p \) is the cross-sectional area of a single piston which is instantaneously exposed to the cylinder-bore pressure, \( P_n \). From the geometry shown in Fig. 3, the position of the nth piston may be written as

\[ x_n = r \tan(\theta_n) \sin(\theta_n) \]  

(14)

Once again, for small swash-plate angles, Eq. (14) may be substituted into Eq. (13) and terms may be rearranged to produce the following result for the reaction force exerted on the swash-plate by the nth piston-slipper assembly

\[ R_n = M_p r \sin(\theta_n) \frac{d^2z}{dt^2} + 2M_p r \cos(\theta_n) \omega \frac{dz}{dt} \]

\[ - M_p r \sin(\theta_n) \omega^2 z + P_n A_p \]  

(15)

where \( \omega \) is the angular velocity of the pump shaft. Note: this result assumes that the mass of the slipper is small compared to the mass of the piston, and that any mechanical retaining-force acting on the slipper is also small. For fixed-clearance designs this force is identically zero. From the geometry in Fig. 3, it is further seen that the moment-arm dimensions shown in Eq. (6) may be written as

\[ L_n = r \sin(\theta_n) \sec(z) \approx L \quad \text{and} \quad I_n = r \sin(\theta_n) \]  

(16)

Substituting Eqs. (9), (12), (15), and (16) into Eq. (6) and rearranging terms produces the following intermediate result for the swash-plate equation-of-motion

\[
\left( I + \frac{M_b + M_c}{2} L^2 + \frac{N}{2} M_p r^2 \right) \frac{d^2x}{dt^2} + C \frac{dx}{dt} \\
+ \left( k L^2 - \frac{N}{2} M_p r^2 \omega^2 \right) x \\
= \left( P_d A_b - P_c A_c \right) L \\
+ F_n L - A_p \sum_{n=1}^{N} P_n \sin(\theta_n)
\]  

(17)

where from the Lagrange trigonometric identities \([10–12]\) it has been recognized that

\[
\sum_{n=1}^{N} \sin^2(\theta_n) = \frac{N}{2} \quad \text{and} \quad \sum_{n=1}^{N} \sin(\theta_n) \cos(\theta_n) = 0
\]  

(18)

This result is only true for typical machine designs in which the pistons are evenly spaced in a circular array about the centerline of the cylinder block. In the following paragraph, the pressure terms on the right-hand-side of Eq. (17) will be evaluated.

As shown in Eq. (17), the swash-plate dynamics depend upon the fluid pressure that is instantaneously generated within the nth piston bore, \( P_n \). In previous research \([7–12]\), this pressure has been studied numerically and has been shown to be fairly constant while the pistons are located directly over the intake and discharge ports, and that the pressure changes almost linearly as the pistons pass over the transition slots on the valve plate. Because this is so common, the fluid pressure within the nth piston bore is usually described using the pressure profile shown in Fig. 5 \([14]\). In this figure the average transition-angle on the valve plate is shown by the symbol \( \gamma \), which is normally called the pressure carryover-angle. Using Fig. 5, the following discrete representation for the fluid pressure within the nth piston bore may be written

\[
P_n = \begin{cases} 
P_d & \frac{\pi}{2} + \gamma < \theta_n < \frac{\pi}{2} \\
\frac{P_d - P_i}{\gamma} (\theta_n - \pi/2) & \frac{\pi}{2} < \theta_n < \frac{\pi}{2} + \gamma \\
P_i & \frac{\pi}{2} + \gamma < \theta_n < \frac{3\pi}{2} \\
\frac{P_i - P_d}{\gamma} (\theta_n - 3\pi/2) & \frac{3\pi}{2} < \theta_n < \frac{3\pi}{2} + \gamma 
\end{cases}
\]  

(19)

It can be shown that when this result is substituted into Eq. (17) it produces a saw-tooth pressure term that oscillates at a frequency of twice piston-pass frequency. For a nine-piston pump running at
2,000 rpm, the frequency of this oscillation is 600 Hz. Since this frequency is far above the bandwidth frequency for anything being controlled in the system, only an average effect of this term is needed to capture its overall dynamic-impact on the system. Therefore, using Eq. (19) a piecewise integral-average may be taken to show that

\[
\sum_{n=1}^{N} P_n \sin(\theta_n) \to \frac{N}{2\pi} \int_{0}^{2\pi} P_n \sin(\theta_n) \, d\theta_n = \frac{N}{2\pi} (P_f - P_i) \gamma
\]

(20)

where the result has been linearized for small values of \( \gamma \).

To summarize the analysis for the swash-plate dynamics, Eq. (20) is substituted into Eq. (17) to produce the following result

\[
\left( I + [M_b + M_c L^2 + \frac{N}{2} M_p r^2] \right) \frac{d^2 \varphi}{dt^2} + C \frac{dx}{dt} + \left( k L^2 - \frac{N}{2} M_p r^2 \alpha^2 \right) (x - x_o) = (A_b - \frac{A_o}{2}) (P_d - P_o) - A_c (P_i - P_d/2)
\]

(21)

Where from steady-state analysis it has been recognized that the bias-spring preload exerts the following moment about the pivot axis of the swash plate

\[
F_oL = \left( k L^2 - \frac{N}{2} M_p r^2 \alpha^2 \right) x_o + \frac{N A_o r \gamma}{2\pi} (P_d - P_i)
\]

\[- \left( A_b - \frac{A_o}{2} \right) LP_d_d
\]

(22)

and the nominal or steady-state actuator pressure is given by half the steady-state discharge pressure (a common design-assumption which will be used throughout this research).

1.5.4 Control Pressure. Equation (21) shows that the dynamics of the swash-plate are partially determined by the fluid pressure within the control actuator. Figure 6 shows a schematic of this actuator where the fluid volume within the actuator is shown by \( V_c \), the volumetric flow-rate into the actuator is shown by \( Q_c \), and the displacement of the actuator is shown by \( x_c \). Based upon similar assumptions that were used to derive Eq. (5), the fluid pressure within the control actuator may be modeled with the following equation

\[
\frac{V_c \, dP_c}{\beta} = Q_c + A_c L \frac{dx_c}{dt}
\]

(23)

Here it is assumed that the absolute change in the actuator volume is small and that the motion of the actuator is governed by Eq. (11) for small swash-plate angles. The volumetric flow-rate into the actuator is provided by the three-way control valve shown in Fig. 1. This valve will be analyzed in the following paragraphs.

1.5.5 Three-Way Control Valve. Figure 7 shows the threeway valve that is used to provide volumetric flow into the pump control-actuator. This valve is shown to be comprised of a housing with an inside spool that slides horizontally in the \( \zeta \)-direction. The left-hand-side of the spool is exposed to the discharge pressure of the pump, which may be thought of as the control feedback-signal. (Recall that the objective of this system is to control the discharge pressure of the pump.) The right-hand-side of the spool is connected to a stiff spring with a mechanical spring-rate shown by the symbol \( k_c \). The spool is shown to move against the spring in the positive \( \zeta \)-direction until a force equilibrium is achieved. The supply pressure to the valve is the discharge pressure of the pump and the return line is generally connected to a reservoir at atmospheric pressure. The volumetric flow rate into the pump actuator is shown in Fig. 7 by taking the difference of the volumetric flow-rates in and out of the valve. Mathematically, this is expressed as

\[ Q_c = Q_1 - Q_2 \]

(24)

It is customary to model the volumetric flow-rate across a hydraulic control valve using the classical orifice-equation which assumes steady, incompressible and inviscid flow conditions. While these assumptions are not strictly adhered to in the case of this analysis, the use of the classical orifice-equation is justified based upon its common utility in research that is similar [1–15] and the mathematical expediency that this equation offers. Using the classical orifice-equation the volumetric flow-rate in an out of the valve may be written as

\[ Q_1 = A_1 C_d \sqrt{\frac{2}{\rho} (P_d - P_r)} \]

\[ Q_2 = A_2 C_d \sqrt{\frac{2}{\rho} (P_c - P_r)} \]

(25)

where \( A_1 \) and \( A_2 \) are the open flow-areas on either side of the central metering-land of the spool valve, \( C_d \) is the discharge coefficient that is assumed to be the same on both sides of the valve, and \( \rho \) is the fluid mass-density. The expressions in Eq. (25) may be linearized as follows

\[ Q_1 = \frac{1}{2} Q_o + k_q \xi + k_c (P_d - P_r) \]

\[ Q_2 = \frac{1}{2} Q_o - k_q \xi + k_c (P_c - P_r) \]

(26)

where \( Q_o \) is the nominal flow rate across each passage, \( \xi \) is the displacement of the three-way valve shown in Fig. 7, and the nominal pressure-drop across each flow passage is taken to be half the nominal discharge pressure. The new coefficients in Eq. (26) are commonly called the flow gain and the pressure-flow coefficient [13,14] given respectively as

\[ K_q = C_d \sqrt{\frac{p}{\rho} \left| \frac{\partial A_{o,2}}{\partial \xi} \right|} \]

\[ K_c = C_d A_o \sqrt{\rho P_o} \]

(27)

where it is assumed that the magnitude of the area gradient on each side of the valve is equal and where \( A_o \) is the nominal openarea of the open-centered valve when the spool is centered in a neutral and nominal position. Note: the open-centered valve is shown in Fig. 7 by the dimension \( u \), and it is assumed that the valve motion \( \xi \) never exceeds this dimension. In other words, the valve moves very little and never fully closes off the open flow-area on either side of the valve. By setting the return-line pressure, \( P_r \), equal to zero, Eqs. (24) and (26) may be used to express the volumetric flow-rate into the control actuator as

\[ Q_c = 2K_q \xi + K_c (P_d - 2P_r) \]

(28)
From this equation it may be seen that the volumetric flow-rate into the control actuator is automatically adjusted by moving the valve in the positive \( n \)-direction. In the following paragraphs the equation-of-motion for the valve spool will be derived.

The bottom image in Fig. 7 illustrates the free-body-diagram of the valve spool. As previously mentioned, the forces acting on the spool include the discharge pressure force on the left-hand-side of the spool and the spring force on the right-hand-side of the spool. Although it is less obvious, there are also substantial fluid-momentum forces acting on the valve. These forces are traditionally called “flow forces” and are shown in the bottom image of Fig. 7 by the symbols \( F_1 \) and \( F_2 \). Summing forces in the positive \( n \)-direction and setting them equal to the time-rate-of-change of linear momentum for the spool, the following equation may be generally written to describe the motion of the spool

\[
M \frac{d^2 \xi}{dt^2} = P_d A_v - F_{wp} - k_s \xi + F_1 + F_2 \tag{29}
\]

where \( A_v \) is the cross-sectional area of the spool, and \( F_{wp} \) is the preload on the spring when the valve is in a perfectly centered and nominal position (i.e., when \( \xi = 0 \)). Before this equation can be of much use an accurate expression for the flow forces must be determined.

The central image of Fig. 7 shows two control volumes of fluid that are found within the three-way spool valve. These control volumes are noted in the figure as CV1 and CV2 and are located on the left and right-hand-side of the central metering land, respectively. These control volumes show fluid crossing the control-volume boundaries and within the control volumes fluid must also move in the positive \( \xi \)-direction in order to pass through the valve. The motion of this fluid creates fluid-momentum that must be resisted by external forces acting on each control volume. These forces are shown in Fig. 7 by the symbols \( F_1 \) and \( F_2 \) as they act on each control volume and as they exert an equal-and-opposite force on the spool. Using the Reynolds transport theorem, the fluid momentum-effects in the positive \( \xi \)-direction for CV1 may be written as

\[
-F_1 = \frac{\partial}{\partial t} \left( \int \rho Q_1 \, dA_1 \right) + \int \rho \left( \frac{Q_1}{A_1} \right)^2 \cos(\varphi) \, dA_1
= \rho \lambda \frac{\partial Q_1}{\partial t} + \rho \frac{Q_1^2}{A_1} \cos(\varphi)
\tag{30}
\]

where the linear dimension \( \lambda \) is shown in the top image of Fig. 7 and describes the region of fluid within the control volume that is moving in the positive \( \xi \)-direction. The jet angle \( \varphi \) is also illustrated in Fig. 7 and has been shown by Richard von Mises (1883–1953) to be approximately 69° for most valve openings [13,14]. In a similar way, the fluid momentum-effects in the positive \( \xi \)-direction for CV2 may be written as

\[
-F_2 = \frac{\partial}{\partial t} \left( \int \rho Q_2 \, dA_2 \right) - \int \rho \left( \frac{Q_2}{A_2} \right)^2 \cos(\varphi) \, dA_2
= \rho \lambda \frac{\partial Q_2}{\partial t} - \rho \frac{Q_2^2}{A_2} \cos(\varphi)
\tag{31}
\]

Substituting the flow models of Eq. (26) into Eqs. (30) and (31), the net flow-force acting on the spool may be written as

\[
F_1 + F_2 = -\rho \lambda K_r \frac{dp_d}{dt} - K_p (p_d - 2p_c) - 2 K_p \xi
\tag{32}
\]

where the pressure-flow coefficient \( K_r \) is shown in Eq. (27) and the flow-force gain and the pressure-flow force coefficient [14] are given respectively as
\[ K_{\phi} = P_d \cdot C_d^2 \cos(\phi) \cdot \left| \frac{\partial A_{1,2}}{\partial \xi} \right| \big|_{\text{in}} \quad \text{and} \quad K_{\phi} = 2A_cC_d^2 \cos(\phi) \]  

(33)

In this equation, as in Eq. (27), it is assumed that the magnitude of the area gradient on each side of the valve is equal and \( A_c \) is the nominal open-area of the open-centered valve when the spool is centered in a neutral and nominal position. In Eq. (32), it can be seen that the steady flow-forces acting on the valve contribute to the stiffness of the valve while both the steady and transient effects introduce pressure disturbances. It is also worth noting that the transient damping effect that is normally anticipated from the flow force has vanished due to the symmetry of the operating conditions and the valve design. Substituting Eq. (32) into Eq. (29) produces the following result for the equation-of-motion for the spool

\[ M \frac{d^2 \xi}{dt^2} + (k_s + 2K_{\phi}) \xi = (P_d - P_a)A_s - K_{\phi\xi} (P_d - 2P_e) - \rho \dot{\xi} K_{\phi\xi} \frac{dP_d}{dt} \]  

(34)

where it has been recognized from the steady-state-operating conditions that \( F_{sp} = P_dA_s \).

2 Summary

To summarize the work of this analysis, and to prepare for the physical analysis which follows, it will be convenient to nondimensionalize the governing equations using the following definitions

\[ P_d = P_d \bar{P}_d, \quad P_e = \frac{P_e}{2} \bar{P}_e, \quad \eta = \omega \bar{\omega}, \quad \xi = \bar{\xi}, \quad t = \bar{t}, \quad \text{and} \quad \tau = \frac{\bar{\xi}}{\bar{\eta}} \]  

(35)

where the parameters with carets are dimensionless and of the order one. The characteristic time \( \tau \) has been chosen in this form to fully capture the dynamics of the discharge pressure, which is the parameter being controlled by the pump. Using this equation with the previous analysis the governing equations may be written in nondimensional form as follows

\[ \Psi_1 = \frac{K_{\phi\xi}}{G_p \bar{\eta}}, \quad \Psi_2 = 2 \left( \frac{L + [M_b + M_r L^2 + 2 \frac{N}{2} M_p r^2]}{A_c L P_d \bar{\eta}} \right) \bar{\omega}, \quad \Psi_3 = 2 \left( \frac{A_b}{A_c} \frac{N A_p r^3}{2 \pi A_c L} \right), \quad \Psi_6 = \frac{V_c P_d}{4 \bar{\xi} K_{\phi\xi} \bar{\eta}}, \]  

\[ \Psi_4 = \frac{2 \left( k L^2 - \frac{N}{2} M_p r^3 \bar{\eta} \right)}{A_c L P_d \bar{\eta}}, \quad \Psi_5 = 2 \left( \frac{A_b}{A_c} \frac{N A_p r^3}{2 \pi A_c L} \right), \quad \Psi_9 = \frac{M_p \bar{\omega}}{A_c P_d \bar{\eta}}, \quad \Psi_{10} = \frac{(k + 2K_{\phi\xi}) \bar{\omega}}{A_c P_{d,\xi}}, \quad \Psi_{11} = \frac{K_{\phi\xi}}{A_c}, \quad \text{and} \quad \Psi_{12} = \frac{\rho \dot{\xi} K_{\phi\xi}}{A_c \tau} \]  

Each of the nondimensional groups carries a physical meaning that is described in Table 1 along with the numerical value that is calculated for that group using the operating and design parameters listed in the Appendix. The characteristic time for this system is shown in Eq. (35) and calculated using information in the Appendix as \( \tau = 0.102 \) s.

2.1 Reduced-Order Model. The immediate benefit to be realized by nondimensionalizing the governing equations is found in our ability to reduce the order of the model by neglecting terms that are small, as identified by small values for specific nondimensional groups. Table 1 shows that for the swash-plate equation, the swash-plate inertia term and the viscous-drag term may be safely neglected because \( \Psi_3 \) and \( \Psi_5 \) are extremely small compared to other nondimensional groups in this equation. Similarly, for the control-pressure equation, the pressure-transient term may be safely neglected. For the spool valve equation, the valve inertia and the transient flow-force may be safely neglected. By recognizing these

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Physical description</th>
<th>Numerical value</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \Psi_1 )</td>
<td>Pump leakage</td>
<td>9.85 \times 10^{-2}</td>
</tr>
<tr>
<td>( \Psi_2 )</td>
<td>Swash-plate inertia</td>
<td>1.78 \times 10^{-3}</td>
</tr>
<tr>
<td>( \Psi_3 )</td>
<td>Swash-plate viscous-drag</td>
<td>9.21 \times 10^{-3}</td>
</tr>
<tr>
<td>( \Psi_4 )</td>
<td>Swash-plate spring rate</td>
<td>1.48 \times 10^{-1}</td>
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<tr>
<td>( \Psi_5 )</td>
<td>Discharge pressure moment exerted on the swash plate</td>
<td>5.93 \times 10^{-1}</td>
</tr>
<tr>
<td>( \Psi_6 )</td>
<td>Pressure transient in the control actuator</td>
<td>1.08 \times 10^{-3}</td>
</tr>
<tr>
<td>( \Psi_7 )</td>
<td>Open-centered valve flow into the control actuator</td>
<td>5.00 \times 10^{-1}</td>
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<tr>
<td>( \Psi_8 )</td>
<td>Volumetric change of the control actuator</td>
<td>1.80 \times 10^{-2}</td>
</tr>
<tr>
<td>( \Psi_9 )</td>
<td>Spool valve inertia</td>
<td>1.69 \times 10^{-6}</td>
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<tr>
<td>( \Psi_{10} )</td>
<td>Spool valve spring rate</td>
<td>6.95 \times 10^{-1}</td>
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<tr>
<td>( \Psi_{11} )</td>
<td>Pressure induced flow-force on the spool valve</td>
<td>9.87 \times 10^{-2}</td>
</tr>
<tr>
<td>( \Psi_{12} )</td>
<td>Transient flow force on the spool valve</td>
<td>1.42 \times 10^{-4}</td>
</tr>
</tbody>
</table>
facts, the complexity of the governing equations may be reduced to a second-order model and written in state-space form as follows:

\[
\begin{bmatrix}
\frac{d\hat{P}}{dt} \\
\frac{d\hat{x}}{dt}
\end{bmatrix} = \begin{bmatrix}
-\Psi_1 & 1 \\
-\Xi_1 & -\Xi_2
\end{bmatrix} \begin{bmatrix}
\hat{P} - 1 \\
\hat{x} - 1
\end{bmatrix}
\tag{38}
\]

where \(\Psi_1\) is shown in Eq. (37) and the other matrix elements are given by:

\[
\Xi_1 = 1 + (1 - \Psi_3)(\Psi_7\Psi_{10} - \Psi_{11}) \\
\Xi_2 = \frac{\Psi_4(\Psi_7\Psi_{10} - \Psi_{11})}{\Psi_8\Psi_{10}}
\tag{39}
\]

Figure 8 shows a block diagram of this second-order system for the purposes of illustrating causality. Using the state-space form of Eq. (38), the dimensionless form for the natural-frequency and the damping ratio may be expressed as:

\[
\hat{\omega}_n = \sqrt{\omega_n^2 + \Xi_2} \quad \text{and} \quad \zeta = \frac{\Psi_1 + \Xi_2}{2 \omega_n}
\tag{40}
\]

The bandwidth frequency is defined as the sinusoidal input forcing-frequency at which the output amplitude of the discharge pressure is reduced to \(\sqrt{2}/2\) times the input amplitude of the desired discharge pressure. This result has been derived in previous literature \cite{14} for a second order system and is presented here in dimensionless form as follows:

\[
\hat{\omega}_{bw} = \hat{\omega}_n \left\{ 1 - 2\zeta^2 + \sqrt{2 - 2\zeta^2(1 - \zeta^2)} \right\}^{1/2}
\tag{41}
\]

Again, the carets indicate a dimensionless quantity. To dimensionalize the frequency results, the following definitions should be used:

\[
\omega_n = \omega_{n0} \tau \quad \text{and} \quad \omega_{bw} = \omega_{bw0} \tau
\tag{42}
\]

where the characteristic time \(\tau\) is shown in Eq. (35). Using the design information in the Appendix and the characteristic time given by \(\tau = 0.102\) s, it may be shown that the undamped natural-frequency is given by \(\omega_{n0} = 14.6\) Hz, the damping ratio is given by \(\zeta = 0.152\), and the bandwidth frequency is given by \(\omega_{bw0} = 22.4\) Hz. These are well-known and typical values for pressure controlled, axial-piston pumps that are designed using a construction similar to that of Fig. 1.

2.2 Bandwidth-Frequency Results. The primary objective of this research is to identify the design parameters that have the greatest impact on the bandwidth frequency of the pump. This objective will be pursued by indentifying the dimensionless group that has the greatest impact on the bandwidth frequency and by then focusing on the design parameters that constitute this dimensionless group. To investigate the sensitivity of the bandwidth frequency to small perturbations of a dimensionless group, a Taylor series expansion of the bandwidth frequency as it has been presented in Eq. (41) will be performed as follows:

\[
\hat{\omega}_{bw} = \hat{\omega}_{bw0} + \frac{\partial \hat{\omega}_{bw}}{\partial \Psi_i_4} |_{o} (\Psi_1 - \Psi_4) + \frac{\partial \hat{\omega}_{bw}}{\partial \Psi_4} |_{o} (\Psi_4 - \Psi_4_4) \\
+ \cdots + \frac{\partial \hat{\omega}_{bw}}{\partial \Psi_{11}} |_{o} (\Psi_{11} - \Psi_{11})
\tag{43}
\]

where the partial differentials are the sensitivity coefficients that correspond to a particular perturbation of a dimensionless group. Note: these partial differentials are evaluated using the nominal operating-conditions and design parameters that are listed in the Appendix. Because these differentials are complex in their analytical form, they will only be presented here in numerical form having been evaluated using the information in the Appendix.

From Table 2, it may be seen that the largest sensitivity to a small perturbation is the one associated with the volumetric change in the control actuator. Furthermore, it may be seen that this sensitivity coefficient is at least 2 orders of magnitude larger than any of the other coefficients listed in the table which illustrates the near-singular importance of this dimensionless group. The negative sign on this coefficient indicates that as the dimensionless group \(\Psi_8\) increases, the bandwidth frequency will decrease. Similarly, as this dimensionless group decreases, the bandwidth frequency will increase. The dimensionless group \(\Psi_8\) is shown in Eq. (37) and is rewritten here for convenience:

\[
\Psi_8 = \frac{A_1 L \zeta}{2 K_{a2} \tau}
\tag{44}
\]

<table>
<thead>
<tr>
<th>Table 2</th>
<th>Bandwidth-frequency sensitivity coefficients</th>
</tr>
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<tbody>
<tr>
<td>Coefficient</td>
<td>Physical description</td>
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<tr>
<td>(\frac{\partial \hat{\omega}<em>{bw}}{\partial \Psi</em>{7}</td>
<td>_{0}})</td>
</tr>
<tr>
<td>(\frac{\partial \hat{\omega}<em>{bw}}{\partial \Psi</em>{10}</td>
<td>_{0}})</td>
</tr>
<tr>
<td>(\frac{\partial \hat{\omega}<em>{bw}}{\partial \Psi</em>{11}</td>
<td>_{0}})</td>
</tr>
<tr>
<td>(\frac{\partial \hat{\omega}<em>{bw}}{\partial \Psi</em>{12}</td>
<td>_{0}})</td>
</tr>
<tr>
<td>(\frac{\partial \hat{\omega}<em>{bw}}{\partial \Psi</em>{13}</td>
<td>_{0}})</td>
</tr>
<tr>
<td>(\frac{\partial \hat{\omega}<em>{bw}}{\partial \Psi</em>{14}</td>
<td>_{0}})</td>
</tr>
<tr>
<td>(\frac{\partial \hat{\omega}<em>{bw}}{\partial \Psi</em>{15}</td>
<td>_{0}})</td>
</tr>
</tbody>
</table>

---

*Pressure-controlled pumps tend to behave as second-order systems while displacement controlled pumps often behave as first-order systems \cite{14}.*
From this expression it may be seen that the frequency response of the pump may be most effectively increased by reducing the swept volume of the actuator, $A_1$, $L_x$, or by increasing the amount of flow that can be delivered to the actuator by the valve, $2K_p u$. Physically, this result says that the frequency response of the pump is primarily limited by how fast the moving actuator can be filled and emptied by the three-way valve. Thus, the assumptions that the actuator can be filled and emptied quickly then the frequency response will be high. Although other parameter adjustments can be made to influence the frequency response of the pump, the point of this analysis has been to show that these are the most significant changes that can be made, and that small adjustments to the actuator area, the actuator moment-arm, and the valve flow-gain will have the greatest impact on the bandwidth frequency of the pump. From the analysis presented here, it can be shown that a 10% change in the bandwidth frequency, which would be quite significant.

As it turns out, the actuator area and the moment-arm are typically designed to overcome the steady-state torque that is exerted on the swash-plate by the pumping-elements of the machine. See Eq. (22). Obviously, this design consideration has nothing to do with the bandwidth frequency of the pump and therefore these parameters cannot be changed without regard to the steady-state operating needs of the machine. As shown in Eq. (22), the most effective way to reduce the required actuator area $A_1$ and the moment-arm $L_x$ is to decrease the pressure carry-over angle $\gamma$ by carefully designing the valve plate shown in Fig. 5.

As shown by Eq. (44), the bandwidth frequency of the pump may be increased by increasing the flow-gain of the valve. The flow gain was presented in Eq. (27) and is rewritten here for convenience

$$K_p = C_d \sqrt{\frac{P_{\text{dc}}}{\rho} \frac{\partial A_1}{\partial \xi} \bigg|_{\xi=0}}$$

(45)

From this equation it can be seen that the primary design-feature for the flow gain is the area gradient for the flow passage. Because circular drill-bits are typically used to machine these flow passages, the open flow area tends to be a crescent-shaped passage that can be mathematically expressed as

$$A_1 = \frac{\pi}{8} d^2 - \frac{d}{3} - (u + \xi) \sqrt{d - u - (\xi)(u + \xi)}$$

$$+ \frac{1}{4} d^2 \sin^{-1} \left( \frac{2u + \xi}{d} - 1 \right)$$

(46)

where $d$ is the diameter of the drill bit that was used to make the flow passage, $u$ is the nominal opening of the flow passage, and $\xi$ is the motion of the spool valve from its perfectly centered position. See Fig. 7 for these dimensions. From this equation the area gradient and nominal area-gradient are given, respectively, as

$$\frac{\partial A_1}{\partial \xi} = 2 \sqrt{(d - u + \xi)(u + \xi)} \quad \frac{\partial A_1}{\partial \xi} \bigg|_{\xi=0} = 2 \sqrt{(d - u)} u$$

(47)

where the nominal area-gradient has been evaluated at $\xi = 0$. The conclusion to be drawn from Eq. (47) is that the area gradient for a circular-shaped flow passage may be increased by increasing the diameter of the flow passage and by increasing the nominal opening of the flow passage. By doing one or both of these things the bandwidth frequency of the pump may be increased.

3 Conclusions

The analysis and bandwidth-frequency results of this paper support the following conclusions:

(1) That the analytical model for a pressure controlled, axial-piston pump involves the consideration of the downstream pressure transients, the movement of the swash plate, the dynamics of the control actuator, and the motion and flow characteristics of the three-way control valve.

(2) That the reaction forces between the piston-slipper assemblies within the pump, and the swash-plate are complex, and that the these forces may be simplified using integral averaging techniques and the Lagrange trigonometric identities [10–12].

(3) That the flow characteristics of the three-way valve may be simplified by considering the linearized concepts of the valve flow-gain, and the valve pressure-flow coefficient.

(4) That the Reynolds Transport Theorem may be used to derive the net flow-force acting on the three-way spool valve, and that these forces induce both steady and transient forces on the valve.

(5) That the steady flow-forces acting on the valve introduce an additional spring-type force that is proportional to valve displacement along with a steady pressure-disturbance force.

(6) That the transient flow-force acting on the valve introduces pressure disturbance effect but that the normally expected damping effect from this force vanishes due to design and operating-condition symmetry.

(7) That the method of nondimensionalizing the governing equations for this system is useful for gaining insight into the dominant physics of the problem, and that this insight is useful for reducing the complexity of analysis. For example, using nondimensional analysis the sixth-order dynamical system has been reduced to a second-order system.

(8) That the swash-plate inertia and viscous-damping effects are small in comparison to the pressure and spring-type moments that are exerted on the swash plate.

(9) That the pressure transient effects within the control actuator are negligible compared to the volumetric flow-rates in and out of the actuator, and the dynamic change of the actuator volume itself.

(10) That the spool valve inertia and pressure-transient flow force acting on the spool are insignificant compared to the pressure and spring-type forces acting on the valve.

(11) That the reduced second-order model may be used to write analytical expressions for the pump’s undamped natural-frequency, the damping ratio, and the bandwidth frequency that produce typical and well-known results for this system.

(12) That perturbation analysis may be conducted for studying the sensitivity of the bandwidth frequency to small changes in design parameters, and that the dimensionless group which describes volumetric changes in the control actuator tends to dominate the physical characteristics of the bandwidth frequency.

(13) That, by far, the most effective way to increase the bandwidth frequency of the pump is to reduce the swept volume of the control actuator and to increase the flow gain of the three-way valve. Physically speaking, this conclusion states that the frequency response of the pump is primarily limited by how fast the moving actuator can be filled and emptied by the three-way valve.

(14) That since the bandwidth frequency can be increased by increasing the flow gain of the valve, this can be accomplished for circular flow-passages by increasing the size of the valve flow-passage and by increasing the nominal opening of the open-centered valve when the spool is in its centered position.

In the process of seeking to understand the physical limitations for increasing the bandwidth frequency of the pressure controlled, axial-piston pump, this paper has presented the most comprehensive dynamical-model for this machine that has appeared in the literature to date. Not only has this paper considered the complex governing-equation for the swash-plate motion, but it has also presented the governing equation for the spool valve motion which includes the steady and transient flow-forces that are exerted on the
valve itself. These complexities have been added to the analysis of this research because they have often been suspected as an unknown cause for limiting the dynamical performance of the pressure controlled, axial-piston pump. As shown in this research, these complexities are not the limiting cause of the machine’s dynamical performance and the dynamical behavior of the pump may be modified primarily by adjusting the actuator swept-volume and the flow capacity of the three-way control valve.

Acknowledgment

The authors would like to thank Caterpillar, Inc., for the generous support of this project, and for the many useful conversations that have taken place during the development of results.

Nomenclature

<table>
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<th>Symbol</th>
<th>Description</th>
<th>Value</th>
<th>Units</th>
</tr>
</thead>
<tbody>
<tr>
<td>( A_b )</td>
<td>cross-sectional area of the bias actuator</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( A_c )</td>
<td>cross-sectional area of the control actuator</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( A_s )</td>
<td>steady-state open flow-area on the right and left-hand-side of the spool valve</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( A_p )</td>
<td>cross-sectional area of a single piston within the pump</td>
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<tr>
<td>( A_k )</td>
<td>open flow-area on the left-hand-side of the spool valve</td>
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<tr>
<td>( A_2 )</td>
<td>open flow-area on the right-hand-side of the spool valve</td>
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<td>( C )</td>
<td>effective viscous-drag coefficient for the swash plate</td>
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<td>valve discharge-coefficient</td>
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<td>( F_{p} )</td>
<td>force exerted on the swash-plate by the control actuator</td>
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<td>flow force exerted on the spool valve from the left-hand control volume (Fig. 7)</td>
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<td>( k_{b} )</td>
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<td>mass of the bias actuator</td>
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<td>mass of the control actuator</td>
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<td>( \frac{M_{p}}{A_{p}} )</td>
<td>mass of a single piston within the pump</td>
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<td>( M_{s} )</td>
<td>mass of the spool valve</td>
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<td>number of pistons within the pump</td>
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<td>counter for the nth piston-slipper assembly</td>
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<td>control pressure</td>
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<td>instantaneous pressure within the nth piston chamber</td>
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<td>( \frac{Q_{p}}{A_{p}} )</td>
<td>volumetric flow-rate from the pump</td>
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<tr>
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Operating conditions

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<thead>
<tr>
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Pump and discharge-line design parameters

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Value</th>
<th>Units</th>
</tr>
</thead>
<tbody>
<tr>
<td>( A_{b} )</td>
<td>78.5</td>
<td>mm²</td>
</tr>
<tr>
<td>( A_{c} )</td>
<td>154</td>
<td>mm²</td>
</tr>
<tr>
<td>( A_{p} )</td>
<td>244</td>
<td>mm²</td>
</tr>
<tr>
<td>( C )</td>
<td>0.60</td>
<td>N m s</td>
</tr>
<tr>
<td>( F_{s} )</td>
<td>854</td>
<td>N</td>
</tr>
<tr>
<td>( G_{p} )</td>
<td>4.62 × 10⁻³</td>
<td>lpm/deg</td>
</tr>
<tr>
<td>( I )</td>
<td>1.14 × 10⁻²</td>
<td>kg m²</td>
</tr>
<tr>
<td>( k )</td>
<td>25</td>
<td>N/mm</td>
</tr>
<tr>
<td>( L )</td>
<td>65</td>
<td>mm</td>
</tr>
<tr>
<td>( M_{b} )</td>
<td>0.012</td>
<td>kg</td>
</tr>
<tr>
<td>( M_{c} )</td>
<td>0.024</td>
<td>kg</td>
</tr>
<tr>
<td>( M_{p} )</td>
<td>0.058</td>
<td>kg</td>
</tr>
<tr>
<td>( \omega_{n} )</td>
<td>3 liters</td>
<td></td>
</tr>
<tr>
<td>( V_{d} )</td>
<td>50</td>
<td>cc/rev</td>
</tr>
<tr>
<td>( V_{1} )</td>
<td>9.00 × 10⁻³</td>
<td>liters</td>
</tr>
<tr>
<td>( V_{2} )</td>
<td>3.70</td>
<td>liters</td>
</tr>
<tr>
<td>( K )</td>
<td>3.57 × 10⁻¹²</td>
<td>m³/(Pa s)</td>
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</tbody>
</table>
Valve design-parameters

<table>
<thead>
<tr>
<th>Description</th>
<th>Symbol</th>
<th>Value</th>
<th>Units</th>
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</thead>
<tbody>
<tr>
<td>Valve area</td>
<td>$A_v$</td>
<td>12.57</td>
<td>mm²</td>
</tr>
<tr>
<td>Valve spring-rate</td>
<td>$k_r$</td>
<td>100</td>
<td>N / mm</td>
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<tr>
<td>Valve mass</td>
<td>$M_v$</td>
<td>$2.96 \times 10^{-3}$</td>
<td>kg</td>
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<tr>
<td>Underlapped valve-dimension</td>
<td>$u$</td>
<td>1.5</td>
<td>mm</td>
</tr>
<tr>
<td>Pressure-flow coefficient</td>
<td>$K_c$</td>
<td>$2.14 \times 10^{11}$</td>
<td>m³/(Pa s)</td>
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<tr>
<td>Pressure flow-force coefficient</td>
<td>$K_{fc}$</td>
<td>$1.24 \times 10^{-6}$</td>
<td>N/Pa</td>
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<tr>
<td>Flow gain</td>
<td>$K_q$</td>
<td>$2.85 \times 10^{-1}$</td>
<td>m³/s</td>
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<tr>
<td>Distance between ports</td>
<td>$\lambda$</td>
<td>10</td>
<td>mm</td>
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</table>

References


