Mixed-strategy Equilibria in a Quality Differentiation Model*

X. Henry Wang
Department of Economics
University of Missouri
Columbia, MO 65211, USA

and

Bill Z. Yang
Division of Social Science
Minot State University
Minot, ND 58707, USA

* We wish to thank Professor Simon Anderson and two anonymous referees for valuable suggestions.

Abstract

This paper shows that the standard quality-differentiation duopoly model has, in addition to the two well-known pure-strategy equilibria of maximum quality differentiation, an infinity of mixed-strategy equilibria in which firms choose mixed strategies in the first-stage quality game. In these equilibria, maximum quality differentiation does not occur due to coordination failure. Total expected consumer surplus is the same at all mixed-strategy equilibria and is higher than that under either pure-strategy equilibrium. Total expected industry profit is the same at all mixed-strategy equilibria and is lower than that under either pure-strategy equilibrium.

JEL Classification Numbers: C70, D40.

Key Words: quality-price game, mixed-strategy equilibrium.

1. Introduction

Standard quality-differentiation duopoly models (e.g., Tirole, 1988; Anderson, de Palma, and Thisse, 1992) predict that maximum quality differentiation results from a two-stage quality-price game. However, casual empiricism suggests that competitors frequently choose very similar qualities for their products. By adopting mixed strategies in quality choice, this paper shows that firms may indeed choose similar qualities in a two-stage quality-price game. [1]

The literature of two-stage quality-price games has generally been concerned with pure-strategy equilibria. A coordination device that has been utilized in the analysis is that one firm always chooses a lower quality than the other firm. However, no reason has been offered as to why any firm is willing to choose a lower quality, especially in light of the fact that the high-quality firm reaps a larger profit in equilibrium.[2] Absent of any coordination device, one would expect that firms adopt similar strategies in quality choice. By allowing for mixed strategies, this paper shows that there is indeed a symmetric mixed-strategy equilibrium in which the two firms adopt the same mixed strategy in their quality choice. In this equilibrium, the probability that the two firms choose the same quality is greater than one half. It is also shown that there is an infinity of mixed-strategy equilibria in which the firms choose mixed strategies in the first-stage quality game. Total expected consumer surplus is the same at all mixed-strategy equilibria and is higher than the total consumer under either pure-strategy equilibrium. Total expected industry profit is also the same at all mixed-strategy equilibria and is lower than the total industry profit under either pure-strategy equilibrium.

---

[1] Mixed-strategy equilibria are reasonable when firms choose independently without any knowledge of each other's choice and when decisions are irreversible. Bester et al. (1996) provide a study of mixed-strategy equilibria in Hotelling's model of horizontal differentiation. They also provide some reasons as to why mixed strategies are reasonable to consider in the two-stage location-price game.

[2] One possible explanation is that firms move sequentially and the firm moving first chooses the higher quality. But then the question becomes: which firm moves first?

2. Model and Results

For simplicity, we adopt the quality-price game in Tirole (1988, p. 296).[3] There are two firms, 1 and 2, that play a two-stage noncooperative game. In the first stage of the game, the firms choose their qualities, s_1 and s_2, in the interval $[\underline{s}, \bar{s}]$ ($\bar{s} > \underline{s} > 0$). In the second stage, they choose their prices, p_1 and p_2. For simplicity, both firms are assumed to have zero marginal production costs. There is a continuum of consumers, indexed by their preference indicator θ which is uniformly distributed over the interval $[\underline{\theta}, \bar{\theta}]$ satisfying $\underline{\theta} > 0$ and $\bar{\theta} = 1 + \underline{\theta}$. A consumer of type θ either buys one unit of the good from one of the two firms or does not buy at all; his/her utility is given by $\theta s - p$ when a unit of quality s is bought at price p, and is zero if no purchase is made. It is assumed that $2\underline{\theta} \le \bar{\theta} \le (\underline{s} + 2\bar{s})\underline{\theta}$. This assumption ensures that all consumers are served in the second-stage price game under any quality configuration.[4]

It is well known that the second-stage price game has a unique Nash equilibrium given any quality pair (s_1, s_2). In particular, firm 1's price is[5]

$$\begin{aligned} p_1 &= (\bar{\theta} - 2\underline{\theta})(s_2 - s_1)/3 \qquad \text{if } s_1 \le s_2, \\ &= (2\bar{\theta} - \underline{\theta})(s_1 - s_2)/3 \qquad \text{if } s_1 > s_2; \end{aligned} \tag{1}$$

its sales are

$$\begin{aligned} D_1 &= (\bar{\theta} - 2\underline{\theta})/3 \qquad \text{if } s_1 < s_2, \\ &= (2\bar{\theta} - \underline{\theta})/3 \qquad \text{if } s_1 > s_2; \end{aligned} \tag{2}$$

---

[3] Pioneering research on vertical product differentiation was done by Gabszewicz and Thisse (1979) and Shaked and Sutton (1982). Tirole (1988) introduces a simplified model that leads to simpler solutions. Tirole's simplified model has since become the model of choice for much of the subsequent research, e.g., see Ronnen (1991), Choi and Shin (1992), Wauthy (1996), and Lehmann-Grube (1997).

[4] If this assumption is not satisfied, Choi and Shin (1992) show that the two firms may choose less than maximum quality differentiation in equilibrium. One can also apply the mixed-strategy framework in the present paper to Choi and Shin's model.

[5] Results in (1)-(3) can be found in Tirole (1988, pp. 296-7). It is worthwhile to note that the second-stage price equilibrium is unique, even when mixed strategies in prices are also considered.

and its profit is

$$\pi_1 = (\bar{\theta} - 2\underline{\theta})^2 (s_2 - s_1)/9 \qquad \text{if } s_1 \le s_2,$$

$$= (2\bar{\theta} - \underline{\theta})^2 (s_1 - s_2)/9 \qquad \text{if } s_1 > s_2. \qquad (3)$$

Firm 2's price, sales, and profit in the second-stage equilibrium are similarly defined. Equations (1)-(3) show that the high-quality firm charges a higher price, obtains larger sales, and reaps a greater profit. Note that from (3) each firm wants to choose a quality as far away from the other firm's quality as possible, given the other firm's quality choice.

It is well known that the two-stage game has two pure-strategy Nash equilibria in the first stage (these are subgame perfect Nash equilibria for the whole game): firm 1 chooses $\underline{s}$ and firm 2 chooses $\bar{s}$, or firm 1 chooses $\bar{s}$ and firm 2 chooses $\underline{s}$, both implying maximum quality differentiation. In these equilibria, neither firm randomizes in quality choice. The natural question is: which firm will choose the high quality? This is a coordination problem. We tackle this problem in the following by considering mixed-strategy subgame perfect Nash equilibria.

Mixed-Strategy Equilibria

Recall that the second-stage price equilibrium is unique given any quality pair. As a result, mixed strategies are possible only in quality choices. Our focus will therefore be on the firms' first stage quality choices. We begin by examining the firms' best responses in quality to a mixed strategy in quality by the other firm.

Without loss of generality, we focus on firm 1's optimal quality responses. Symmetry between the two firms implies that any conclusion about firm 1 applies to firm 2 as well. Suppose firm 2 chooses its quality according the probability distribution function $F(s_2)$. From (3), firm 1's expected profit

$$E\pi_1(s_1) = [\int_{\underline{s}}^{s_1} (s_1 - s_2) dF(s_2)](2\bar{\theta} - \underline{\theta})^2 / 9 + [\int_{s_1}^{\bar{s}} (s_2 - s_1) dF(s_2)](\bar{\theta} - 2\underline{\theta})^2 / 9. \qquad (4)$$

In (4), the distribution function F(.) is in a general form; it may correspond to a discrete distribution, a continuous distribution, or a mixture of those. Based on (4), we obtain the following lemma about firm 1's optimal quality responses to a mixed strategy in quality by firm 2.

Lemma 1. (i) Firm 1's optimal quality response is $\underline{s}$ or $\bar{s}$ or the whole interval $[\underline{s}, \bar{s}]$ when firm 2 randomizes between $\underline{s}$ and $\bar{s}$.

(ii) Firm 1's optimal quality response is $\underline{s}$ and/or $\bar{s}$ for any mixed strategy by firm 2 except when firm 2 randomizes between $\underline{s}$ and $\bar{s}$.

Proof: Consider part (i) first. Using (3), if firm 2 randomizes between $\underline{s}$ and $\bar{s}$ with probabilities λ and $1-\lambda$ then firm 1's expected profit is

$$E\pi_1(s_1) = [-\frac{(2\bar{\theta}-\underline{\theta})^2}{9}\lambda\underline{s} + \frac{(\bar{\theta}-2\underline{\theta})^2}{9}(1-\lambda)\bar{s}\ ] + [\frac{(2\bar{\theta}-\underline{\theta})^2}{9}\lambda - \frac{(\bar{\theta}-2\underline{\theta})^2}{9}(1-\lambda)\ ]\, s_1 .$$

Hence, if $(2\bar{\theta}-\underline{\theta})^2\lambda < (\bar{\theta}-2\underline{\theta})^2(1-\lambda)$ firm 1's optimal choice is $\underline{s}$; if $(2\bar{\theta}-\underline{\theta})^2\lambda > (\bar{\theta}-2\underline{\theta})^2(1-\lambda)$ firm 1's optimal choice is $\bar{s}$; and if

$$(2\bar{\theta}-\underline{\theta})^2\lambda = (\bar{\theta}-2\underline{\theta})^2(1-\lambda) \qquad (5)$$

then firm 1 is indifferent between all values of s_1 in the interval $[\underline{s}, \bar{s}]$, i.e., firm 1's optimal choice is the whole interval $[\underline{s}, \bar{s}]$. This proves part (i) of the lemma.

Now consider part (ii). We will focus our attention on two types of mixed strategies by firm 2: discrete and continuous probability density functions.[6] First, suppose firm 2 randomizes by choosing s_2^1, s_2^2, …, s_2^n, … with respective probabilities λ_1, λ_2, …, λ_n, …, where $\underline{s} \le s_2^1 \le s_2^2 \le \ldots \le s_2^n \le \ldots \le \bar{s}$,

[6] For the more general case where firm 2's mixed strategy in quality is given by a probability density function that is composed of discrete and continuous segments, the same general shape holds for firm 1's expected profit function and therefore the same conclusion as obtained below holds. While our proof of Proposition 1 is restricted to a (wide and natural) class of possible mixed strategies, we believe it generalizes to all possible mixed strategies.

$\lambda_i \geq 0$ for all i and $\lambda_1 + \lambda_2 + \ldots = 1$.[7] Since any monotonic sequence from a compact interval must converge, for notational convenience we will include in the sequence the limiting value of the sequence $\{s_2^n\}$ and denote it by s_2^∞.[8] Denote $\underline{s}$ by s_2^0 and $\bar{s}$ by $s_2^{\infty+1}$. For any s_1 in the interval $[s_2^i, s_2^{i+1}]$ (i = 0,1,…,∞), firm 1's expected profit in (4) takes the form[9]

$$E\pi_1(s_1) = \frac{(2\bar{\theta}-\underline{\theta})^2}{9}[\lambda_1(s_1 - s_2^1) + \ldots + \lambda_i(s_1 - s_2^i)] + \frac{(\bar{\theta}-2\underline{\theta})^2}{9}[\lambda_{i+1}(s_2^{i+1} - s_1) + \lambda_{i+2}(s_2^{i+2} - s_1) + \ldots]$$

$$= [-\frac{(2\bar{\theta}-\underline{\theta})^2}{9}(\lambda_1 s_2^1 + \ldots + \lambda_i s_2^i) + \frac{(\bar{\theta}-2\underline{\theta})^2}{9}(\lambda_{i+1}s_2^{i+1} + \lambda_{i+2}s_2^{i+2} + \ldots)]$$

$$+ [\frac{(2\bar{\theta}-\underline{\theta})^2}{9}(\lambda_1 + \ldots + \lambda_i) - \frac{(\bar{\theta}-2\underline{\theta})^2}{9}(\lambda_{i+1} + \lambda_{i+2} + \ldots)]s_1. \quad (6)$$

The expression (6) shows three facts about the function $E\pi_1(s_1)$. First, $E\pi_1(s_1)$ is a linear function of s_1 on each subinterval $[s_2^i, s_2^{i+1}]$. Second, $E\pi_1(s_1)$ is continuous in s_1 on the whole interval $[\underline{s}, \bar{s}]$.[10] Third, the coefficient of s_1 in (6) increases as s_1 moves across subintervals from left to right.[11] These facts imply that $E\pi_1(s_1)$ is a piece-wise linear and continuous function of s_1 (with rising slopes across subintervals from left to right). Further, they imply that firm 1's expected profit function takes three possible shapes: (a) $E\pi_1(s_1)$ always increases in s_1, if $s_2^1 = \underline{s}$ and λ_1 is sufficiently large; (b) it always

---

[7] This formulation includes as a special case where firm 2 randomizes over only a finite number of qualities, this can be done by equating all s_2^i from a point onwards.

[8] The proof generalizes to the more general case where the support of the mixed strategy consists of multiple and possibly non-monotonic convergent sequences. One interesting example is $\{s_2^i\}_{i=1}^\infty$ with $s_2^i = s + (-1)^i/(i+1)$, which converges oscillatingly to $s \in (\underline{s}, \bar{s})$.

[9] Note that both infinite sums in (6) exist due to the fact that they are bounded and monotonic in i.

[10] To show that $E\pi_1(s_1)$ is continuous in s_1, one need only verify that it is continuous at $s_1 = s_2^1, s_2^2, \ldots, s_2^\infty$. The latter follows from the fact that (6) implies $E\pi_1(s_1)$ is the same value whether $s_1 = s_2^i$ is regarded as the right end of the interval $[s_2^{i-1}, s_2^i]$ or the left end of the interval $[s_2^i, s_2^{i+1}]$.

[11] Note that the first bracketed term on the RHS of the second equality of (6) moves in the opposite direction from that of the coefficient of s_1 on the RHS of the second equality of (6).

decreases in s_1, if $s_2^\infty = \bar{s}$ and λ_∞ is sufficiently large; (c) otherwise, it exhibits a "U" shape that when s_1 is close to $\underline{s}$, the coefficient of s_1 is negative and hence $E\pi_1(s_1)$ decreases in s_1, while when s_1 is close to $\bar{s}$, the coefficient of s_1 is positive and hence $E\pi_1(s_1)$ increases in s_1. Therefore, firm 1's optimal choice of s_1 is $\bar{s}$ in case (a) and $\underline{s}$ in case (b). In case (c), firm 1's optimal choice of s_1 is $\underline{s}$ or $\bar{s}$, or both.

Next, suppose firm 2 randomizes according to a continuous density function $f(s_2)$. In this case, firm 1's expected profit in (4) becomes

$$E\pi_1(s_1) = [\int_{\underline{s}}^{s_1} (s_1 - s_2) f(s_2) ds_2](2\bar{\theta} - \underline{\theta})^2/9 + [\int_{s_1}^{\bar{s}} (s_2 - s_1) f(s_2) ds_2](\bar{\theta} - 2\underline{\theta})^2/9 . \qquad (7)$$

Applying the Leibneiz' rule, the derivative of (7) with respect to s_1 is

$$[\int_{\underline{s}}^{s_1} f(s_2) ds_2](2\bar{\theta} - \underline{\theta})^2/9 - [\int_{s_1}^{\bar{s}} f(s_2) ds_2](\bar{\theta} - 2\underline{\theta})^2/9 ,$$

which is negative for small s_1 and positive for large s_1. Hence, firm 1's optimal choice is $\underline{s}$ and/or $\bar{s}$. We have thus proved part (ii) of the lemma.

It follows from Lemma 1 that in any mixed-strategy equilibrium at least one firm must randomize exclusively between $\underline{s}$ and $\bar{s}$. Suppose firm 2 does not randomize exclusively between $\underline{s}$ and $\bar{s}$ in a mixed-strategy equilibrium. Then firm 1 must optimally respond by choosing either $\underline{s}$ or $\bar{s}$ or both, by part (ii) of Lemma 1. If firm 1 chooses an end point as a pure strategy then it cannot be a mixed-strategy equilibrium, since firm 2's optimal choice is the other end point leading to a pure-strategy equilibrium. Hence, firm 1 must randomize exclusively between $\underline{s}$ and $\bar{s}$.

We now examine the mixed-strategy equilibria in which firm 1 randomizes exclusively between $\underline{s}$ and $\bar{s}$. Note that a firm must be indifferent among all the qualities over which it randomizes in a mixed-strategy equilibrium. Therefore, in a mixed-strategy equilibrium where firm 1 randomizes exclusively between $\underline{s}$ and $\bar{s}$, firm 2's equilibrium strategy $F(s_2)$ must be such that firm 1's expected profit given by (4) is equal at $\underline{s}$ and at $\bar{s}$. That is,

$$[\int_{\underline{s}}^{\bar{s}}(\bar{s}-s_2)dF(s_2)](2\bar{\theta}-\underline{\theta})^2/9 = [\int_{\underline{s}}^{\bar{s}}(s_2-\underline{s})dF(s_2)](\bar{\theta}-2\underline{\theta})^2/9.$$

Define $s^* = \int_{\underline{s}}^{\bar{s}} s_2 dF(s_2)$, the expected quality of firm 2 under the mixed strategy F(s_2), and solve for it from the above equation, we have

$$s^* = \frac{(\bar{\theta}-2\underline{\theta})^2\underline{s}+(2\bar{\theta}-\underline{\theta})^2\bar{s}}{(\bar{\theta}-2\underline{\theta})^2+(2\bar{\theta}-\underline{\theta})^2}, \tag{8}$$

which lies in the interval ($\underline{s}$, $\bar{s}$), implying that F(s_2) cannot be a pure strategy in s_2 at either $\underline{s}$ or $\bar{s}$.

Likewise, in an equilibrium where firm 1 randomizes exclusively between $\underline{s}$ and $\bar{s}$ with probabilities λ and 1-λ, respectively, for firm 2 to adopt the equilibrium strategy F(s_2) that satisfies (7), the probability λ must be such that firm 2 is indifferent among the qualities over which it randomizes. It means, by Lemma 1, that equation (5) must be satisfied, since otherwise firm 2 would either choose $\underline{s}$ or choose $\bar{s}$ as a pure strategy. Solving for λ from equation (5) yields

$$\lambda^* = \frac{(\bar{\theta}-2\underline{\theta})^2}{(\bar{\theta}-2\underline{\theta})^2+(2\bar{\theta}-\underline{\theta})^2}, \tag{9}$$

a value between 0 and 1/2. It is worth noting that $\lambda^*\underline{s}+(1-\lambda^*)\bar{s}=s^*$. Hence, the mean quality of both firms is s^* in any mixed-strategy equilibrium.

We summarize the results on mixed-strategy equilibrium in the following proposition.

Proposition 1. There are two classes of mixed-strategy equilibria in quality choice. One class is where firm 1 randomizes exclusively between $\underline{s}$ and $\bar{s}$ with probabilities λ^* and $1-\lambda^*$, respectively, and firm 2 randomizes over the interval [$\underline{s}$, $\bar{s}$] according to any probability distribution function F(s_2) with the expected quality s^*.[12] The other class is obtained by switching the two firms' choices.

Two special cases of interest are stated in the following corollary, as implied from the above Proposition.[13]

Corollary 1. (i) There is a unique symmetric mixed-strategy equilibrium wherein both firms choose $\underline{s}$ with probability λ^* and $\bar{s}$ with probability $1-\lambda^*$.

(ii) There are two mixed-strategy equilibria in which only one firm adopts a mixed strategy. One equilibrium is where firm 1 randomizes exclusively between $\underline{s}$ and $\bar{s}$ with probability λ^* and $1-\lambda^*$, respectively, and firm 2 adopts a pure-strategy choice at s^*; the other equilibrium is symmetric to the first one by switching the two firms' choices.

Comparison of Pure- and Mixed-Strategy Equilibria

We next compare pure and mixed-strategy equilibria. More specifically, we examine total consumer surplus, profits and the expected difference in equilibrium qualities.

Obviously, the two pure-strategy equilibria yield the same total consumer surplus, profits, and quality difference. Consider the pure-strategy equilibrium where $s_1 = \underline{s}$ and $s_2 = \bar{s}$. In this equilibrium, profits are (superscript "ps" denotes for pure strategy)

$$\pi_1^{ps} = (\bar{\theta} - 2\underline{\theta})^2 (\bar{s} - \underline{s})/9 \quad \text{and} \quad \pi_2^{ps} = (2\bar{\theta} - \underline{\theta})^2 (\bar{s} - \underline{s})/9, \tag{10}$$

total consumer surplus is

[12] In particular, if firm 2 randomizes over $\{\underline{s}, s^*, \bar{s}\}$ with probabilities $\{\alpha\lambda^*, 1-\alpha, \alpha(1-\lambda^*)\}$ the expected quality under this mixed strategy is s^* for any $\alpha \in [0, 1]$.

[13] The two cases in Corollary 1 correspond respectively to $\alpha = 1$ and $\alpha = 0$ in the previous footnote.

$$CS^{ps} = \int_{\underline{\theta}}^{(\bar{\theta}+\underline{\theta})/3}[\theta \underline{s} - \frac{\bar{\theta}-2\underline{\theta}}{3}(\bar{s}-\underline{s})]d\theta + \int_{(\bar{\theta}+\underline{\theta})/3}^{\bar{\theta}}[\theta \bar{s} - \frac{2\bar{\theta}-\underline{\theta}}{3}(\bar{s}-\underline{s})]d\theta$$

$$= [(\frac{\bar{\theta}+\underline{\theta}}{3})^2 - \underline{\theta}^2]\underline{s}/2 - (\frac{\bar{\theta}-2\underline{\theta}}{3})^2(\bar{s}-\underline{s}) + [\bar{\theta}^2 - (\frac{\bar{\theta}+\underline{\theta}}{3})^2]\bar{s}/2 - (\frac{2\bar{\theta}-\underline{\theta}}{3})^2(\bar{s}-\underline{s}), \quad (11)$$

total industry profit is

$$\Pi^{ps} = (\bar{\theta}-2\underline{\theta})^2(\bar{s}-\underline{s})/9 + (2\bar{\theta}-\underline{\theta})^2(\bar{s}-\underline{s})/9, \quad (12)$$

and quality difference is $\bar{s}-\underline{s}$.

Now consider mixed-strategy equilibria. Obviously, given the symmetry between the two classes of mixed-strategy equilibria, we need only consider one class for the purpose of comparison. Consider the class of mixed-strategy equilibria in which firm 1 randomizes between $\underline{s}$ and $\bar{s}$ with probabilities λ^* and $1-\lambda^*$ respectively, and firm 2 randomizes over the interval [$\underline{s}$, $\bar{s}$] according to a probability distribution function $F(s_2)$ with the expected quality s^*. For any given quality choice of s_2 by firm 2, if firm 1 randomizes between $\underline{s}$ and $\bar{s}$ with probabilities λ^* and $1-\lambda^*$ respectively then the firms' profits are

$$E\pi_1 = \lambda^*(s_2-\underline{s})(\bar{\theta}-2\underline{\theta})^2/9 + (1-\lambda^*)(\bar{s}-s_2)(2\bar{\theta}-\underline{\theta})^2/9 \quad \text{and} \quad (13)$$

$$E\pi_2 = \lambda^*(s_2-\underline{s})(2\bar{\theta}-\underline{\theta})^2/9 + (1-\lambda^*)(\bar{s}-s_2)(\bar{\theta}-2\underline{\theta})^2/9, \quad (14)$$

total expected consumer surplus is

$$\lambda^*\left\{\int_{\underline{\theta}}^{(\bar{\theta}+\underline{\theta})/3}[\theta \underline{s} - \frac{\bar{\theta}-2\underline{\theta}}{3}(s_2-\underline{s})]d\theta + \int_{(\bar{\theta}+\underline{\theta})/3}^{\bar{\theta}}[\theta s_2 - \frac{2\bar{\theta}-\underline{\theta}}{3}(s_2-\underline{s})]d\theta\right\}$$

$$+ (1-\lambda^*)\left\{\int_{\underline{\theta}}^{(\bar{\theta}+\underline{\theta})/3}[\theta s_2 - \frac{\bar{\theta}-2\underline{\theta}}{3}(\bar{s}-s_2)]d\theta + \int_{(\bar{\theta}+\underline{\theta})/3}^{\bar{\theta}}[\theta \bar{s} - \frac{2\bar{\theta}-\underline{\theta}}{3}(\bar{s}-s_2)]d\theta\right\}, \quad (15)$$

total expected industry profit is

$$\lambda^*(s_2-\underline{s})\frac{(\bar{\theta}-2\underline{\theta})^2+(2\bar{\theta}-\underline{\theta})^2}{9} + (1-\lambda^*)(\bar{s}-s_2)\frac{(\bar{\theta}-2\underline{\theta})^2+(2\bar{\theta}-\underline{\theta})^2}{9}, \quad (16)$$

and expected quality difference is

$$\lambda^*(s_2-\underline{s}) + (1-\lambda^*)(\bar{s}-s_2). \quad (17)$$

The corresponding equilibrium values are obtained by taking expectations of these expressions with respect to s_2. Since all expressions in (13)-(17) are linear in firm 2's quality s_2, it is implied that the corresponding equilibrium values are obtained by simply substituting s_2 in the above expressions by the expected value of s_2, s^*. Hence, in equilibrium, firms' profits are (superscript "ms" denotes for mixed strategy)

$$E\pi_1^{ms} = E\pi_2^{ms} = \lambda^*(1-\lambda^*)[(\bar{\theta}-2\underline{\theta})^2 + (2\bar{\theta}-\underline{\theta})^2]/9, \qquad (18)$$

total expected consumer surplus is

$$CS^{ms} = \lambda^*\left\{\int_{\underline{\theta}}^{(\bar{\theta}+\underline{\theta})/3}[\theta\underline{s} - \frac{\bar{\theta}-2\underline{\theta}}{3}(s^*-\underline{s})]d\theta + \int_{(\bar{\theta}+\underline{\theta})/3}^{\bar{\theta}}[\theta s^* - \frac{2\bar{\theta}-\underline{\theta}}{3}(s^*-\underline{s})]d\theta\right\}$$

$$+ (1-\lambda^*)\left\{\int_{\underline{\theta}}^{(\bar{\theta}+\underline{\theta})/3}[\theta s^* - \frac{\bar{\theta}-2\underline{\theta}}{3}(\bar{s}-s^*)]d\theta + \int_{(\bar{\theta}+\underline{\theta})/3}^{\bar{\theta}}[\theta\bar{s} - \frac{2\bar{\theta}-\underline{\theta}}{3}(\bar{s}-s^*)]d\theta\right\}$$

$$= 2\lambda^*(1-\lambda^*)[CS^{ps}] + (\lambda^*)^2(\bar{\theta}^2-\underline{\theta}^2)\underline{s}/2 + (1-\lambda^*)^2(\bar{\theta}^2-\underline{\theta}^2)\bar{s}/2, \qquad (19)$$

total expected industry profit is

$$\Pi^{ms} = \lambda^*(s^*-\underline{s})\frac{(\bar{\theta}-2\underline{\theta})^2 + (2\bar{\theta}-\underline{\theta})^2}{9} + (1-\lambda^*)(\bar{s}-s^*)\frac{(\bar{\theta}-2\underline{\theta})^2 + (2\bar{\theta}-\underline{\theta})^2}{9}$$

$$= 2\lambda^*(1-\lambda^*)\Pi^{ps}, \qquad (20)$$

and expected quality difference is[14]

$$2\lambda^*(1-\lambda^*)(\bar{s}-\underline{s}). \qquad (21)$$

Results in (18) show that both firms obtain the same expected profit at all mixed-strategy equilibria.[15] (19)-(21) indicate that total expected consumer surplus, total expected industry profit and expected quality difference are all the same at all mixed-strategy equilibria. We summarize these results

---

[14] Note that the difference in expected quality is zero since both firms have the same expected quality in equilibrium.

[15] Comparing the firms' expected profits given by (18) with their profits given by (10) and using λ^* in (9), one finds that both firms earn less in a mixed-strategy equilibrium than in either of the pure-strategy equilibria. This is a typical property of coordination games, such as the well-known "Battle of the Sexes" game, in which both players' payoffs in the mixed-strategy equilibrium are less than those under either of the pure-strategy equilibria.

in the following proposition.

Proposition 2. (i) Both firms earn the same expected profit at any mixed-strategy equilibrium.

(ii) Total expected consumer surplus, total expected industry profit, and expected quality difference are the same at all mixed-strategy equilibria.

We next compare the outcomes under pure and mixed-strategy equilibria. Taking the difference between CS^{ms} in (19) and CS^{ps} in (11) to obtain

$$CS^{ms} - CS^{ps} = \{[\beta^2 + (1-\beta)^2](11\delta^2 - 14\delta + 11)/9 - \beta^2\delta^2 - (1-\beta)^2\}(\bar{s} - \underline{s})\underline{\theta}^2 / 2, \quad (22)$$

where $\delta = \bar{\theta} / \underline{\theta}$ and $\beta = \lambda^* / \underline{\theta}$. From (9), it is obvious that β depends only on δ. Hence, the sign of $CS^{ms} - CS^{ps}$ depends only on δ. The assumptions that $\underline{\theta} > 0$ and $2\underline{\theta} \le \bar{\theta}$ imply $\delta \ge 2$. Straightforward calculations show that the term in braces in (22) is positive for all $\delta \ge 2$. Hence, $CS^{ms} > CS^{ps}$ for all $\underline{\theta}$ and $\bar{\theta}$ satisfying the assumptions placed on them in the model setup. That is, total expected consumer surplus is higher under any mixed-strategy equilibrium than under any pure-strategy equilibrium. Since $2\lambda^*(1-\lambda^*) < 1/2$,[16] (20) shows that $\Pi^{ms} < \Pi^{ps}/2$, and (21) shows that the expected quality difference in a mixed-strategy equilibrium is less than half of that in a pure-strategy equilibrium. Thus, we have the following proposition about the comparison between pure and mixed-strategy equilibria.

Proposition 3. (i) Total expected consumer surplus is higher under any mixed-strategy equilibrium than under either pure-strategy equilibrium.

(ii) Total expected industry profit is lower under any mixed-strategy equilibrium than under either pure-strategy equilibrium.

---

[16] This follows from the fact that the maximum value for $2\lambda(1-\lambda)$ is $1/2$, which occurs at $\lambda = 1/2$, and that $\lambda^* < 1/2$.

(iii) Expected quality difference in any mixed-strategy equilibrium is less than half of that under either pure-strategy equilibria.

We provide the following intuition for the above conclusions by further examining the symmetric mixed-strategy equilibrium in which both firms choose $\underline{s}$ with probability λ^* and $\bar{s}$ with probability $1-\lambda^*$. If quality realizations are $\bar{s}$ for both firms then all consumers are better off in the mixed-strategy equilibrium than in a pure-strategy equilibrium since they get the best quality and pay a zero price. If quality realizations are $\underline{s}$ for both firms then most consumers are still better off than under a pure-strategy equilibrium since they pay a zero price. However, the highest valuation consumers may be worse off in this case since zero price may not compensate for their getting a lower-quality product. Total consumer surplus is higher when the latter group is either non-existent or relatively small. Of course, if quality realizations are $\underline{s}$ and $\bar{s}$ the outcome is the same as that under a pure-strategy equilibrium. Proposition 3 shows the expected outcome is better for consumers as a group in the mixed-strategy equilibrium. Finally, firms are worse off in the mixed-strategy equilibrium than in a pure-strategy equilibrium since there is assumed a coordination of quality choices in a pure-strategy equilibrium and none exists in a mixed-strategy equilibrium.

Finally, we note that if no coordination mechanism is in place for quality selections then it seems the focal point among all the pure and mixed-strategy equilibria for the two-stage quality-price game is the symmetric equilibrium in which the two firms adopt the same mixed strategy in their quality choice. Note also that in the symmetric mixed-strategy equilibrium the probability that the firms end up choosing the same quality is $(\lambda^*)^2+(1-\lambda^*)^2$, which is greater than 1/2.

3. Concluding Remarks

This paper has studied the standard quality differentiation duopoly model a la Tirole (1988). It is shown that the two-stage quality-price game has, in addition to the two well-known subgame perfect pure-strategy equilibria of maximum quality differentiation, two classes of subgame perfect mixed-strategy equilibria of less than maximum quality differentiation. In particular, there is a symmetric mixed-strategy equilibrium in which the probability that the two firms choose the same quality is greater than one half.

In concluding this paper, we provide a brief comparison of mixed-strategy equilibria in the quality differentiation model to mixed-strategy equilibria in the Hotelling location model with quadratic transportation costs as studied by Bester el al. (1996). These two models are structurally very similar, the former is a model of vertical product differentiation while the latter is one of horizontal product differentiation. Both papers address the same question: what are the implications of dropping the assumption that the firms' coordination problem is solved exogenously (by restricting to pure strategies). In both models, if firms play pure strategies then there is a desire to be as far away from the other firm as possible, leading to maximum differentiation in pure-strategy equilibria. Both models possess an infinity of mixed-strategy equilibria of less than maximum differentiation. In any mixed-strategy equilibrium in the quality differentiation model, at least one firm randomizes only between the two extreme qualities while in the horizontal differentiation model both firms may randomize among an arbitrary number of different locations in a mixed-strategy equilibrium. Both models have a unique symmetric mixed-strategy equilibrium. In the former, the symmetric mixed-strategy equilibrium involves each firm randomizing only between the two extreme qualities, while in the latter it involves both firms randomizing at all different locations. Overall, equilibrium mixed strategies in the quality differentiation model are less complex than those in the location model.[17]

The main reason for this difference in complexity of mixed strategy lies in the nature of consumer preferences in the two models. In the quality differentiation model consumers are assumed to be homogeneous in the sense that their preferences agree on "higher quality is better"; in the horizontal

[17] Because of the simplicity in equilibrium mixed strategies, welfare analysis is possible in the quality-differentiation model as shown in this paper.

differentiation model consumers are heterogeneous in that each consumer has a different ideal product (or location). This difference in consumer tastes has consequences for the coordination problem of firms. The uni-directed tastes in the quality differentiation model makes the coordination problem of firms more focused towards the boundary points. As a result, the full-coordination qualities in the pure-strategy equilibria remain very prominent in mixed-strategy equilibria, as shown in Proposition 1 of this paper. However, the multi-directed tastes in the horizontal differentiation model makes the coordination problem of firms more severe. As a result, the full-coordination locations in the pure-strategy equilibria are only given minor prominence in mixed-strategy equilibria. Technically, in the quality differentiation model, a firm's expected profit function is generally a U-shaped function of its own quality variable (as shown by equations (6) and (7) above), implying at most two local maxima. However, in the location model, a firm's expected profit function is a piece-wise concave function with possibly many local maxima as shown in the proof of Proposition 3 in Bester at al. These shapes of the expected profit functions are responsible for Proposition 1 in this paper and Propositions 3 and 4 in Bester et al. on mixed-strategy equilibria in the two models.

References

Anderson, Simon, Andre de Palma, and Jacques-Francois Thisse (1992). Discrete Choice Theory of Product Differentiation. Cambridge, Massachusetts: MIT Press.

Bester, Helmut, Andre de Palma, Wolfgang Leininger, Jonathan Thomas, and Ernst-Ludwig von Thadden (1996). "A noncooperative analysis of Hotelling's location game," Games and Economic Behavior, 12, 165-186.

Choi, Chong Ju, and Hyun Song Shin (1992). "A comment on a model of vertical product differentiation," Journal of Industrial Economics, 60, 229-231.

Gabszewicz, Jean and Jacqes-Francois Thisse (1979). "Price competition, quality and income disparities," Journal of Economic Theory, 20, 340-359.

Lehmann-Grube, Ulrich (1997). "Strategic choice of quality when quality is costly: the persistence of the high-quality advantage," Rand Journal of Economics, 28, 372-284.

Ronnen, Uri (1991). "Minimum quality standards, fixed costs, and competition," Rand Journal of Economics, 22, 490-504.

Tirole, Jean (1988). The Theory of Industrial Organization. Cambridge, Massachusetts: MIT Press.

Wauthy, Xavier (1996). "Quality choice in models of vertical differentiation," Journal of Industrial Economics, 64, 345-353.