Duality and consumption decisions under income and price risk

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Abstract

This paper uses duality to provide an analysis of compensated and uncompensated decisions under income and/or price risk. We focus on a two-good model, in which the individual maximizes expected utility subject to a stochastic budget constraint. A Slutsky equation is derived that decomposes the derivative of the optimal decision function with respect to any parameter in the model into an income effect and a substitution effect. The total effect of an increase in risk is controlled by the generalized preference intensity, while the compensated effect of an increase in risk is controlled by the generalized risk premium developed in the literature on aversion to one risk in the presence of another.

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1. Introduction

The two-good model in which an economic agent maximizes expected utility subject to a budget constraint that is stochastic due to endowment (income) risk and/or price (interest rate) risk has been used extensively in the literature on decision making under risk. Beginning with seminal papers on savings decisions under income risk and interest rate risk by Leland (1968), Sandmo (1970), Dreze and Modigliani (1972), the two-good model has been used to study consumption and portfolio choice (Aura et al., 2002), the labor supply decision
under wage risk (Coyte, 1986), the provision of public goods (Sandler et al., 1987), the optimal level of social security benefits (Feldstein, 1985), and taxation of labor income and the demand for risky assets (Elemendorf and Kimball, 2000). Recently, the two-good model has been used to study the effects of partial privatization of the Social Security Trust Fund (Diamond and Geanakoplos, 1999; Abel, 2001).

As in the theory of choice under certainty, there has been a great deal of interest in the risk counterparts of compensated decisions and the Slutsky decomposition of the response of optimal choice to parameter variations (Dreze and Modigliani, 1972; Fischer, 1972; Pope and Chavas, 1985; Davis, 1989; Aura et al., 2002). Compensated choices provide the theoretical underpinnings for the analysis of the welfare effects of parameter changes. Since compensated choices are generally unobservable, they must be inferred from uncompensated choices. The Slutsky equation provides the bridge between compensated and uncompensated choices. For example, compensated (Hicksian) demand elasticities can be deduced from uncompensated (Marshallian) demand elasticities using the Slutsky equation.¹

The effect of a change in expected price on decisions under risk, the counterpart of the effect of a change in price under certainty, has been investigated in the literature beginning with Sandmo (1970). The effect of increases in risk on optimal choice has no counterpart in choice theory under certainty. While this effect has been investigated for restricted forms of increases in risk, there has been no systematic analysis of the effects of Rothschild–Stiglitz (RS) increases in risk on optimal uncompensated and compensated decisions in the two-good model. In this paper, we use a dual approach to analyze the uncompensated and compensated effects of RS increases in income and/or price risk. The key to our approach involves the transformation of the stochastic budget constraint into a deterministic budget constraint and the use of the derived utility function. This enables us to analyze choice under risk in a way that is formally analogous to the analysis of choice under certainty. The duality of expected utility maximization and expected expenditure minimization yields a general Slutsky equation that decomposes the derivative of demand with respect to any parameter in the model into an income effect and a substitution effect.

Since RS increases in risk pertain to increments in risk (i.e., there is no risk free situation), the Arrow-Pratt measures of risk aversion are not strong enough to yield determinate comparative static results. We show that for such increases in risk, the risk aversion measures relevant for comparative static analysis are the incremental preference intensity (IPI) and the incremental-risk premium (IRP). The former is an extension of the Friedman-Savage preference intensity risk aversion function and the latter is derived from the partial risk premium formulated by Ross (1981) and Kihlstrom et al. (1981), and which measures aversion to one risk in the presence of another risk. It turns out that the derivative of the indirect utility function with respect to RS risk parameter is IPI, while the derivative of the expected expenditure function with respect to RS risk parameter is IRP. The behavior of IPI controls the effect of RS increases in risk on uncompensated decision, while the behavior of IRP controls the effect of RS increase in risk on compensated decision.

At the theoretical level, our analysis generalizes some seminal results in the literature and presents some new ones. In particular, we show that our results for income risk generalize those obtained by Leland (1968), Sandmo (1970) for uncompensated decisions and Dreze

¹ See Alston et al. (2002) for an example of estimation of compensated demand.
and Modigliani (1972) for compensated decisions. In the case of price risk, our results generalize and reconcile those obtained by Hanson and Menezes (1978) and Dardanoni (1988). At the empirical level, our analysis has implications for the estimation of consumer demand in the presence of income and price risk.

In Section 2, we present a two-good model of consumer choice under income and price risk. The dual expected (excess) expenditure minimization problem is formulated in Section 3. In Section 4, we interpret the derivatives of the indirect utility function and the expected (excess) expenditure function with respect to RS risk parameter as two incremental-risk aversion functions. Our general results are applied to the specific problems of income risk in Section 5, and price risk in Section 6. Section 7 shows that our hypotheses about the behavior of IPI and IRP are governed by established hypotheses about single-attribute utility functions when the two-attribute utility function is additively separable. In Section 8, we discuss extensions of the two-good model to higher dimensions and also the empirical implications of our analysis on consumer demand.

2. Consumer choice under income and price risk

Consider a consumer of two goods \( x_1 \geq 0 \) and \( x_2 \geq 0 \). Let \( p > 0 \) denote the relative price of good 1 in terms of good 2, and \((\omega_1, \omega_2)\) the consumer’s endowment of the two goods. The consumer faces possibly two independent sources of risk, a random endowment of good two \( \tilde{\omega}_2 \geq 0 \) and random relative price \( \tilde{p} > 0 \). We assume that the consumer’s von Neumann–Morgenstern utility function \( u(x_1, x_2) \) is thrice continuously differentiable, strictly concave, and strictly increasing in both \( x_1 \) and \( x_2 \). The consumer chooses \( x_1 \) and \( \tilde{x}_2 \) to solve the problem:

\[
\begin{align*}
\text{Max} & \quad Eu(x_1, \tilde{x}_2) \\
\text{s.t.} & \quad \tilde{p}x_1 + \tilde{x}_2 = \tilde{p}\omega_1 + \tilde{\omega}_2 \\
& \quad x_1 \geq 0, \tilde{x}_2 \geq 0.
\end{align*}
\]

The model can be interpreted as a two period consumption savings model in which the numeraire good is second period consumption. In this interpretation, the price of first period consumption can be random because of a random interest rate or because of random inflation. This model encompasses other decision models as special cases, such as the labor supply problem under wage risk.

The solution to (1) is difficult to analyze because the consumer is choosing a one-dimensional object \( x_1 \) subject to an infinite-dimensional inequality constraint \( x_2 \) must be non-negative in every state. We proceed by transforming (1) into a problem that is an equivalent formulation, involving non-stochastic constraints and the derived (expected) utility function. Let \( \bar{\omega}_2 \) denote the mean of \( \tilde{\omega}_2 \) and \( \omega_2^* = \tilde{\omega}_2 - \bar{\omega}_2 \). Let \( \bar{p} \) denote the mean of \( \tilde{p} \) and \( p^* = \tilde{p} - \bar{p} \). The distribution functions of \( \omega_2^* \) and \( p^* \) are \( F(\omega_2^*, \alpha) \) and \( G(p^*, \beta) \), where \( \alpha \) and \( \beta \) are indices of Rothschild–Stiglitz risk. The parameter \( \alpha \) is zero when there is no endowment risk and the RS riskiness of \( F \) increases as \( \alpha \) increases. Similarly, \( \beta \) is zero when there is no price risk and the RS riskiness of \( G \) increases as \( \beta \) increases. The supports of \( F \) and \( G \) are, respectively, \([a, b]\) and \([c, d]\), where \( a \leq 0, c \leq 0, b \geq 0 \) and \( d \geq 0 \). Let \( \bar{x}_2 \)
denote the expected value of $\tilde{x}_2$ and let $N$ be the set of points $(x_1, \tilde{x}_2)$ which satisfies the non-negativity constraints $x_1 \geq 0$ and $\tilde{x}_2 \geq 0$. To complete our reformulation of (1), consider the derived utility function

$$v(x_1, \tilde{x}_2, \alpha, \beta) = \int_a^b \int_c^d u(x_1, \tilde{x}_2 + p^*(\omega_1 - x_1) + \omega_2^*) \, dG(p^*, \beta) \, dF(\omega_2^*, \alpha)$$

which represents the consumer’s preferences over pairs $(x_1, \tilde{x}_2)$. Since $u$ is strictly concave, thrice continuously differentiable and strictly increasing, $v$ is strictly concave and thrice continuously differentiable. The consumer’s choice problem (1) is equivalent to choosing $(x_1, \tilde{x}_2)$ to solve

$$\max_{(x_1, \tilde{x}_2) \in N} v(x_1, \tilde{x}_2, \alpha, \beta)$$

s.t.

$$\tilde{p} x_1 + \tilde{x}_2 \tilde{p}\omega_1 + \tilde{\omega}_2,$$

where the mean budget constraint is obtained by taking expectations on both sides of the stochastic budget constraint in (1). The reformulation in (3) is identical in its structure to the maximization problem in conventional demand theory. In this version, $N$ plays the role of the consumption set. We will always assume an interior solution, which means that an indifference curve of the derived utility function is tangent to the mean budget constraint somewhere in the interior of the mean budget portion of the set $N$.

The reformulated problem in (3) includes two special cases that have been of particular interest to the literature. The first arises when only income (i.e. endowment) is risky. In that case, $p$ is non-stochastic, so that $c = p^* = d = 0$, $\tilde{p} = \tilde{p}$. The second arises when only the relative price is risky. In that case, $\omega_2$ is non-stochastic, so that $a = \omega_2^* = b = 0$.

The solution to (3), which is unique because the derived utility function $v$ is strictly concave, is a pair of ordinary demand functions, $(x_1(\tilde{p}, \omega_1, \tilde{\omega}_2, \alpha, \beta), \tilde{x}_2(\tilde{p}, \omega_1, \tilde{\omega}_2, \alpha, \beta))$, obtained from the first-order conditions:

$$m(x_1, \tilde{x}_2, \alpha, \beta) = \tilde{p} \quad \text{and} \quad \tilde{p} x_1 + \tilde{x}_2 = \tilde{p}\omega_1 + \tilde{\omega}_2,$$

where

$$m(x_1, \tilde{x}_2, \alpha, \beta) = \frac{v_1(x_1, \tilde{x}_2, \alpha, \beta)}{v_2(x_1, \tilde{x}_2, \alpha, \beta)}$$

is the marginal rate of substitution between $x_1$ and $\tilde{x}_2$ and subscripts of functions denote partial derivatives.

---

2 The non-negativity constraint $\tilde{x}_2 \geq 0$ holds for all realizations of $\omega_2^*$ and $p^*$ if and only if $\tilde{x}_2 + e(\omega_1 - x_1) + a \geq 0$ and $\tilde{x}_2 + d(\omega_1 - x_1) + a \geq 0$.

3 Except where it is necessary to refer to them, we will suppress $\alpha$ and $\beta$ in $v$ and other functions.

4 In the case of price risk or joint income-price risk, $v$ may not be everywhere increasing in $x_1$ and its indifference curves may be U-shaped. Since the consumer’s optimal consumption point is on the mean budget constraint it must be on the decreasing portion of an indifference curve. Accordingly, in this paper we disregard the upward portion of any U-shaped indifference curve of $v$. 
Totally differentiating the conditions in (4) gives
\[
\begin{bmatrix}
m_1 & m_2 \\
\bar{p} & 1
\end{bmatrix}
\begin{bmatrix}
dx_1 \\
d\bar{x}_2
\end{bmatrix}
= 
\begin{bmatrix}
d\bar{p} - m_\alpha d\alpha - m_\beta d\beta \\
(\omega_1 - x_1) d\bar{p} + \bar{p} d\omega_1 + d\bar{\omega}_2
\end{bmatrix}
\] (6)

from which we obtain the comparative statics derivatives
\[
\frac{\partial x_1}{\partial \bar{p}} = \frac{1 - (\omega_1 - x_1)m_2}{D}, \quad \frac{\partial x_1}{\partial s} = -m_s/D, \quad s \in \{\alpha, \beta\},
\]
\[
\frac{\partial x_1}{\partial \omega_1} = -\bar{p} m_2/D, \quad \frac{\partial x_1}{\partial \bar{\omega}_2} = -m_2/D
\] (7)

where
\[
D(x_1, \bar{x}_2) = m_1 - \bar{p} m_2
\] (8)
is the determinant of the matrix in (6) and is negative by the concavity of \( v \).

We now define the indirect (derived) utility function by
\[
V(\bar{p}, \omega_1, \bar{\omega}_2, \alpha, \beta) = v(x_1(\bar{p}, \omega_1, \bar{\omega}_2, \alpha, \beta), \bar{x}_2(\bar{p}, \omega_1, \bar{\omega}_2, \alpha, \beta), \alpha, \beta).
\]

Adapting familiar arguments from conventional demand analysis one can show that \( V \) has properties analogous to the conventional indirect utility function. Specifically, by the envelope theorem, one can establish a version of Roy’s identity, namely the absolute value of the ratio of the partial derivative of \( V \) with respect to \( \bar{p} \) to the partial derivative of \( V \) with respect to \( \bar{\omega}_2 \) is excess demand for good 1, i.e., demand less the amount of the endowment. In the context of this paper, a key property of the indirect utility function is that (by the envelope theorem) its partial derivative with respect to \( s \) (\( s \in \{\alpha, \beta\} \)) is in fact the partial derivative of the derived utility function with respect to \( s \). Namely,
\[
\frac{\partial V(\bar{p}, \omega_1, \bar{\omega}_2, \alpha, \beta)}{\partial s} = \frac{\partial v(x_1, \bar{x}_2, \alpha, \beta)}{\partial s} \quad \text{for} \quad s \in \{\alpha, \beta\},
\] (9)

where \((x_1, \bar{x}_2) = (x_1(\bar{p}, \omega_1, \bar{\omega}_2, \alpha, \beta), \bar{x}_2(\bar{p}, \omega_1, \bar{\omega}_2, \alpha, \beta))\).

3. The dual problem

Our formulation of the consumer’s choice problem permits a straightforward derivation of compensated demand functions under risk. When a consumer chooses a bundle \((x_1, \bar{x}_2)\), the consumer’s expected excess expenditure is \( \bar{p} x_1 + \bar{x}_2 - \bar{p} \omega_1 - \bar{\omega}_2 \); it is the mean amount by which the consumer’s expenditure exceeds the value of the endowment. We obtain compensated demand by minimizing expected excess expenditure subject to an expected utility constraint. That is, the compensated demand functions, \((y_1(\bar{p}, \omega_1, \bar{\omega}_2, \alpha, \beta, \bar{v}), \bar{y}_2(\bar{p}, \omega_1, \bar{\omega}_2, \alpha, \beta, \bar{v}))\), are obtained as solutions to the problem:
\[
\begin{align*}
\min & \quad \bar{p} y_1 + \bar{y}_2 - \bar{p} \omega_1 - \bar{\omega}_2 \\
\text{subject to} & \quad v(y_1, \bar{y}_2, \alpha, \beta) = \bar{v}.
\end{align*}
\] (10)
The solution to (10) is unique because $v$ is strictly concave. The first-order conditions for an interior solution are $m(y_1, \bar{y}_2, \alpha, \beta) = \bar{p}$ and $v(y_1, \bar{y}_2, \alpha, \beta) = \bar{v}$. Totally differentiating these conditions gives

$$
\begin{bmatrix}
m_1 & m_2 \\
v_1 & v_2
\end{bmatrix}
\begin{bmatrix}
dy_1 \\
d\bar{y}_2
\end{bmatrix}
= \begin{bmatrix}
d\bar{p} - m_\alpha d\alpha - m_\beta d\beta \\
d\bar{v} - v_\alpha d\alpha - v_\beta d\beta
\end{bmatrix}.
$$

From this system, we calculate the comparative statics derivatives

$$
\frac{\partial y_1}{\partial \bar{p}} = \frac{1}{D} \frac{\partial y_1}{\partial \bar{v}} = -\frac{m_2}{(v_2 D)} \frac{\partial y_1}{\partial s} = \left(-m_x + \frac{v_x m_2}{v_2}\right) / D \quad s \in \{\alpha, \beta\}
$$

(11)

where $D$ is as given in (8) with $(x_1, \hat{x}_2)$ replaced by $(y_1, \bar{y}_2)$.

We now define the expected (excess) expenditure function that is the risk counterpart of the conventional (excess) expenditure function. It gives the value of the expected excess expenditure that is minimized in the solution to (10). That is

$$
e(\bar{p}, \omega_1, \omega_2, \alpha, \beta, \bar{v}) \equiv \bar{p}y_1(\bar{p}, \alpha, \beta, \bar{v}) + \bar{y}_2(\bar{p}, \alpha, \beta, \bar{v}) - \bar{p}\omega_1 - \bar{\omega}_2.
$$

(12)

Because $x_1$ is the numeraire, this function measures expected excess expenditure in units of good 2. When the value of the function is zero, the consumer’s endowment is just adequate to provide expected utility $\bar{v}$. When the value of the function is positive, it gives the minimum amount by which the value of the endowment bundle (measured in units of good 2) must be increased to provide a sufficient budget to achieve expected utility of $\bar{v}$. When it’s negative, its absolute value is the maximum amount by which the value of the endowment bundle (measured in units of good 2) may be decreased if the consumer is to achieve expected utility of $\bar{v}$.

Adapting familiar arguments from conventional demand analysis one may show that the function $e$ defined by (12) has properties analogous to the conventional expenditure function. Specifically, it is concave in $\bar{p}$ and, by the envelope theorem, one may demonstrate a version of Shephard’s lemma, namely that the partial derivative of $e$ with respect to $\bar{p}$ is compensated excess demand for good 1. Because of its importance for our subsequent analysis of the generalized Slutsky equation, we state the following result relating compensated and ordinary demand formally.

**Lemma 1.**

$$
x_1(\bar{p}, \omega_1, \omega_2 + e(\bar{p}, \omega_1, \omega_2, \alpha, \beta, \bar{v}), \alpha, \beta) = y_1(\bar{p}, \alpha, \beta, \bar{v}),
$$

$$
\hat{x}_2(\bar{p}, \omega_1, \omega_2 + e(\bar{p}, \omega_1, \omega_2, \alpha, \beta, \bar{v}), \alpha, \beta) = \hat{y}_2(\bar{p}, \alpha, \beta, \bar{v}).
$$

**Proof.** In Appendix A. □

The identities in Lemma 1 provide the basis for a general decomposition of the comparative statics of ordinary demand into income and substitution effects. Specifically,
Theorem 1. If \( \bar{v} = v(x_1(\bar{p}, \omega_1, \bar{\omega}_2, \alpha, \beta), \bar{x}_2(\bar{p}, \omega_1, \bar{\omega}_2, \alpha, \beta), \alpha, \beta) \), then
\[
\frac{\partial x_1}{\partial s} = \frac{\partial y_1}{\partial s} - \frac{\partial e}{\partial s} \times \frac{\partial x_1}{\partial \omega_2} \quad \text{and} \quad \frac{\partial x_2}{\partial s} = \frac{\partial \bar{x}_2}{\partial s} - \frac{\partial e}{\partial s} \times \frac{\partial \bar{x}_2}{\partial \omega_2}
\]
for each \( s \in \{ \bar{p}, \alpha, \beta, \bar{\bar{v}} \} \).

Proof. This theorem follows immediately from Lemma 1 by differentiation and the fact that \( e(\bar{p}, \omega_1, \bar{\omega}_2, \alpha, \beta, \bar{\bar{v}}) = 0 \) when \( v(x_1(\bar{p}, \omega_1, \bar{\omega}_2, \alpha, \beta), \bar{x}_2(\bar{p}, \omega_1, \bar{\omega}_2, \alpha, \beta), \alpha, \beta) = \bar{\bar{v}} \).

Theorem 1 provides a Slutsky decomposition of the derivative of ordinary demand with respect to any parameter \( s \in \{ \bar{p}, \alpha, \beta, \bar{\bar{v}} \} \). For each \( s \) the compensated terms \( \frac{\partial y_1}{\partial s} \) and \( \frac{\partial \bar{x}_2}{\partial s} \) are the substitution effects and the terms given by the product \(-[\partial e/\partial s] \times [\partial x_1/\partial \omega_2] \) and \([-[\partial e/\partial s] \times [\partial \bar{x}_2/\partial \omega_2] \) are the income effects. The decomposition in Theorem 1 in principle allows for a “substitution effect” associated with a change in \( \omega_1 \) or \( \bar{\omega}_2 \). However, these substitution effects are degenerate since \( \frac{\partial y_1}{\partial \omega_1} = \frac{\partial y_1}{\partial \bar{\omega}_2} = \frac{\partial \bar{x}_2}{\partial \omega_1} = \frac{\partial \bar{x}_2}{\partial \bar{\omega}_2} = 0 \). The familiar income and substitution effects of conventional demand analysis emerge as a special case of this analysis when there is no risk and when the parameter \( s \) is \( \bar{p} \), which in the absence of risk is simply \( p \). In this case, \( v \) is simply \( u \). Thus, in the absence of risk the Slutsky equation for the total effect of a price change on endowment constrained demand becomes
\[
\frac{\partial x_1}{\partial p} = \frac{\partial y_1}{\partial p} - (x_1 - \alpha_1) \times \frac{\partial x_1}{\partial \omega_2}, \quad \text{where} \quad \frac{\partial e}{\partial p} = x_1 - \alpha_1 \text{ by the envelope theorem.}
\]

Fig. 1 illustrates the Slutsky decomposition in Theorem 1 for a consumer whose only source of risk is income risk. The consumer’s initial choice is the bundle marked \( \text{Old} \) where the indifference curve \( v(x_1, \bar{x}_2, \alpha) = \bar{\bar{v}} \) is tangent to the mean budget line. When the riskiness of income risk increases, the entire indifference map shifts upward because the consumer is risk averse. In the figure, the solid indifference curves represent preferences before risk increases and the dashed indifference curves represent preferences afterwards. If the consumer is compensated for the increase in risk, he will move to the bundle \( \text{New with Compensation} \) where the indifference curve \( v(x_1, \bar{x}_2, \alpha + \Delta \alpha) = \bar{\bar{v}} \) is tangent to the compensated mean budget line. The movement from \( \text{Old} \) to \( \text{New with Compensation} \) is the substitution effect. The movement from \( \text{Old} \) to \( \text{New} \) where the indifference curve \( v(x_1, \bar{x}_2, \alpha + \Delta \alpha) = \bar{\bar{v}}(\leq \bar{\bar{v}}) \) is tangent to the mean budget line is the total effect on demand. The movement from the bundle \( \text{New with Compensation} \) on the compensated mean budget line to the bundle \( \text{New} \) is the income effect. We have drawn Fig. 1 so that both total effect and substitution effect are negative.

4. General comparative statics of increased income and price risk

In this section, we review and extend two aspects of preferences and relate them to the comparative statics of increasing risk. The first is the preference intensity function, a utility-based measure of aversion to risk formulated by Friedman and Savage (1948). This function is relevant for analysis of the introduction of risk into a previously certain environment. Its analog for increases in risk is the incremental preference intensity function. The second is the risk premium associated with an increment in risk in the presence of
Fig. 1. Slutsky decomposition of income risk.

existing risk, which gives the maximum amount that an individual is willing to pay to eliminate the additional risk. We show that the behavior of incremental preference intensity governs the total effect of an increase in risk, and the behavior of incremental-risk premium governs the substitution effect of an increase in risk.

Friedman and Savage (1948) defined the preference intensity $\rho$ as the difference between the utility of the sure option $(x_1, \bar{x}_2)$ and the expected utility of the risky prospect $(x_1, \tilde{x}_2 + x^*_2)$, which has an expected value of $(x_1, \bar{x}_2)$ and interpreted it as a measure of the disutility of risk. Stone (1970) called $\rho$ the generalized risk premium. Formally,

$$\rho(x_1, \tilde{x}_2, \alpha, \beta) = u(x_1, \bar{x}_2) - E[u(x_1, \tilde{x}_2 - \pi + p^x(\omega_1 - x_1) + \omega_2^*)] = u(x_1, \tilde{x}_2) - v(x_1, \tilde{x}_2, \alpha, \beta).$$

It is positive for a risk averter. The analog of this function for an increase in the riskiness of an already risky environment is the disutility of an incremental risk.

**Definition 1.** The partial derivatives of the preference intensity function

$$\rho_s(x_1, \tilde{x}_2, \alpha, \beta) = -\frac{\partial v(x_1, \tilde{x}_2, \alpha, \beta)}{\partial s} \text{ for } s \in \{\alpha, \beta\}$$

are respectively called the *incremental preference intensity (IPI)* for income risk and for price risk.
By (9), \( \rho_s \) is in fact the absolute value of the partial derivative of the indirect utility function with respect to \( s \).

As formulated by Ross (1981), Kihlstrom et al. (1981) and used by Gollier and Pratt (1996) and others, the (partial) risk premium \( \Delta w \) associated with an increase in income risk is defined implicitly by the equation:

\[
\mathbb{E}u(x_1, \tilde{x}_2 + p^*(\omega_1 - x_1) + \omega_2^* + \tilde{\epsilon}) = \mathbb{E}u(x_1, \tilde{x}_2 + p^*(\omega_1 - x_1) + \omega_2^* - \Delta w),
\]

where \( \tilde{\epsilon} \) is an independent increase in income risk, and \( \omega_2^* + \tilde{\epsilon} \) are distributed respectively according to \( F(\omega_2^*, \alpha) \) and \( F(\omega_2^*, \alpha + \Delta \alpha) \). This equation can be rewritten as

\[
v(x_1, \tilde{x}_2, (\alpha + \Delta \alpha), \beta) = v(x_1, \tilde{x}_2 - \Delta w, \alpha, \beta).
\]

In the limit as \( \Delta \alpha \to 0 \), we have \( \lim_{\Delta \alpha \to 0} \Delta w / \Delta \alpha = -v_\alpha / v_2 \). Similarly, the risk premium \( \Delta w \) associated with an increase in price risk is defined implicitly by the equation:

\[
\mathbb{E}u(x_1, \tilde{x}_2 + (p^* + \tilde{\epsilon})(\omega_1 - x_1) + \omega_2^*) = \mathbb{E}u(x_1, \tilde{x}_2 - \Delta w + p^*(\omega_1 - x_1) + \omega_2^*),
\]

where \( p^* \) and \( p^* + \tilde{\epsilon} \) are distributed respectively according to \( G(p^*, \beta) \) and \( G(p^*, \beta + \Delta \beta) \). This equation can be rewritten as

\[
v(x_1, \tilde{x}_2, \alpha, \beta + \Delta \beta) = v(x_1, \tilde{x}_2 - \Delta w, \alpha, \beta).
\]

In the limit as \( \Delta \beta \to 0 \), \( \lim_{\Delta \beta \to 0} \Delta w / \Delta \beta = -v_\beta / v_2 \).

**Definition 2.** Let

\[
\Pi^s \equiv -\frac{v_s}{v_2} \quad \text{for} \quad s \in \{\alpha, \beta\}.
\]

\( \Pi^\alpha \) and \( \Pi^\beta \) are, respectively, called the incremental-risk premium (IRP) for income risk and for price risk.

\( \Pi^\alpha \) (\( \Pi^\beta \)) is the maximum amount of second period income that the individual will pay to eliminate a small increase in income (price) risk. By the envelope theorem, \( \Pi^\alpha \) (\( \Pi^\beta \)) is the partial derivative of the expected expenditure function \( e \) in (12) with respect to \( \alpha \) (\( \beta \)).

We are now in a position to relate the comparative statics of increased income and/or price risk to the behavior of the measures of risk aversion presented in Definitions 1 and 2. Theorem 2 shows that the total effect of increasing risk on demand is governed by the behavior of IPI, while Theorem 3 shows that the substitution effect of increasing risk is governed by the behavior of IRP.

**Theorem 2.** When \( \tilde{v} = v(x_1, \tilde{x}_2, \alpha, \beta) \)

\[
\frac{\partial x_1}{\partial s} = \frac{\partial / \partial x_1 | v = \tilde{v}(x_1, \tilde{x}_2)}{v_2(x_1, \tilde{x}_2)D(x_1, \tilde{x}_2)} \quad \text{for} \quad s = \alpha, \beta.
\]

**Proof.** In the Appendix A. □
Since $D < 0$ and $v_2 > 0$, Theorem 2 indicates that demand for good one decreases (increases) as the RS risk parameter $\alpha$ or $\beta$ increases if and only if IPI for income or price risk increases (decreases) in $x_1$ holding the value of derived utility fixed.

**Theorem 3.**

$$\frac{\partial y_1}{\partial s} = \frac{\partial / \partial y_1 | v = v^{\Pi'(y_1, \tilde{y}_2)}}{D(y_1, \tilde{y}_2)} \text{ for } s = \alpha, \beta.$$  

**Proof.** In the Appendix A \[\square\]

Since $D < 0$, Theorem 3 indicates that compensated demand for good one decreases (increases) as the RS risk parameter $\alpha$ or $\beta$ increases if and only if IRP for income or price risk increases (decreases) in $x_1$ holding the value of derived utility fixed.

5. Income risk

When income risk is the only source of risk, $\beta = 0$. In this situation, IPI in Definition 1 becomes

$$\rho_a = -\int_a^b u(x_1, \tilde{x}_2 + \omega_2^*) f_\alpha(\omega_2^*, \alpha) d\omega_2^* = \int_a^b [-u_{22}(x_1, \tilde{x}_2 + \omega_2^*)]T^F(\omega_2^*, \alpha) d\omega_2^*,$$

where the last equality is obtained by integrating by parts twice. IRP in Definition 2 becomes

$$\Pi^a = \int_a^b \frac{[-u_{22}(x_1, \tilde{x}_2 + \omega_2^*)]T^F(\omega_2^*, \alpha) d\omega_2^*}{E_{\omega_2^*} u_2(x_1, \tilde{x}_2 + \omega_2^*)}$$  

(14)

The following theorem shows that the total effect of an increase in income risk is controlled by the behavior of $-u_{22}$ along an indifference curve and the substitution effect is controlled by the behavior of $-u_{22}/E_{\omega_2^*} u_2$ along an indifference curve.

**Theorem 4.** If the relative price $\tilde{p}$ is risk free then

(a) $\partial x_1 / \partial \alpha < (>) 0$ if and only if $-u_{22}(x_1, \tilde{x}_2)$ increases (decreases) as $x_1$ increases along an indifference curve;

(b) $\partial y_1 / \partial \alpha < (>) 0$ if and only if $-u_{22}(x_1, \tilde{x}_2)/E_{\omega_2^*} u_2(x_1, \tilde{x}_2 + \omega_2^*)$ increases (decreases) as $x_1$ increases along an indifference curve.

**Proof.** In the appendix. \[\square\]

This theorem generalizes several important results in the literature concerning the comparative statics of income risk. The existing literature has shown that the assumption that

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Where $f$ is the p.d.f. associated with $F$, $T^F(t, \alpha) = \int_{\alpha}^t F_\alpha(t, \alpha) dt$ satisfies $T^F(a, \alpha) = T^F(b, \alpha) = 0$ and $T^F(t, \alpha) \geq 0$ for all $t \in [a, b]$. 

absolute risk aversion \( A(x_1, x_2) = -\frac{u_{22}(x_1, x_2)}{u_2(x_1, x_2)} \) increases as \( x_1 \) increases along an indifference curve is sufficient for an increase in (or the introduction of) income risk to decrease good-one consumption (e.g., Leland, 1968; Sandmo, 1970) and is necessary and sufficient for compensated good-one consumption to decrease (e.g., Dreze and Modigliani, 1972). Theorem 4(a) shows that the necessary and sufficient condition for income risk to reduce consumer demand is that \(-u_{22}(x_1, x_2)\) increases in \( x_1 \) along an indifference curve. The significance of \( A(x_1, x_2) \) increasing in \( x_1 \) along an indifference curve is that it is a sufficient condition for \(-u_{22}(x_1, x_2)\) to increase in \( x_1 \) along an indifference curve. The latter condition can be interpreted as saying that the individual becomes more averse to downside risk as \( x_1 \) increases along an indifference curve. Let \((x_1^a, x_2^a)\) and \((x_1^b, x_2^b)\) be two points on an indifference curve with \( x_1^a < x_1^b \). Let \( \bar{\varepsilon} \) be a zero-mean random variable. An individual becomes more averse to downside risk as \( x_1 \) increases along an indifference curve if the individual prefers \((x_1^a, x_2^a + \bar{\varepsilon})\) to \((x_1^b, x_2^b + \bar{\varepsilon})\). That is, the individual prefers the lottery \(\{x_1^a, x_2^a + \bar{\varepsilon}\}, 1/2; (x_1^a, x_2^a)\) to the lottery \(\{x_1^b, x_2^b + \bar{\varepsilon}\}, 1/2; (x_1^b, x_2^b)\). In the latter form, dispersion \( \bar{\varepsilon} \) is transferred downward from \( x_2^a \) to \( x_2^b \). For additively separable preferences, these lotteries reduce to the lotteries described in Section 7 below.

Theorem 4(b) generalizes the result on the substitution effect of income risk by Dreze and Modigliani (1972). Their result pertains to the introduction of income risk to a previously certain environment and is a special case of Theorem 4(b). To see this, note that if there is no income risk initially, i.e., \( \alpha = 0 \) and \( \text{prob}(\omega_2^* = 0) = 1 \), then \( E_{\omega_2^*} u_2(x_1, \bar{x}_2 + \omega_2^*) \) reduces to \( u_2(x_1, \bar{x}_2) \).

6. Price risk

In situations where there is price risk only the derived utility function in (2) becomes

\[
v(x_1, \bar{x}_2) = \int_{c}^{d} u(x_1, \bar{x}_2 + p^*(\omega_1 - x_1)) \, dG.
\]

From the form of \( v \), it can be seen that the consumer can avoid all risks by consuming his endowment, \((x_1, \bar{x}_2) = (\omega_1, \bar{x}_2)\). If \( x_1 - \omega_1 > 0 \), the consumer is a buyer of good 1. If \( x_1 - \omega_1 < 0 \), the consumer is a seller of good 1. The effect of an increase in price risk depends on whether the consumer is a buyer or seller of good 1. In either case, however, the riskiness of period-two consumption, \( \bar{x}_2 + p^*(\omega_1 - x_1) \), grows multiplicatively as \( |x_1 - \omega_1| \) increases.

These features are the key to an understanding of the comparative statics of price risk and are reflected in the properties of the indifference maps. In Fig. 2, the vertical line at \( \omega_1 \) represents riskless consumption bundles \((\omega_1, \bar{x}_2)\). Consider a pair of indifference curves with the same level of expected utility \( \bar{u} \) but correspond to different values of the risk parameter \( \beta \). The solid indifference curve with expected utility \( \bar{u} \) represents a smaller value of \( \beta \) than the dashed indifference curve with the same expected utility \( \bar{u} \). Since the consumer is risk averse, the dashed indifference curve with expected utility \( \bar{u} \) is everywhere above the solid curve except at \( x_1 = \omega_1 \), where they are tangent. To the left of \( \omega_1 \), a high-risk (dashed) indifference curve with a lower expected utility than \( \bar{u} \) will cross the low risk
(solid) indifference curve from above. To the right of $\omega_1$, a high-risk indifference curve with a lower expected utility than $\bar{v}$ will cross the low risk indifference curve from below. We assume that any pair of indifference curves corresponding to different levels of price risk can cross at most twice.

We now discuss the implications of the properties of the indifference map on the behavior of IPI and IRP. When price risk is the only source of risk ($\alpha = 0$), IPI in Definition 1 becomes

$$\rho_\beta = -\int_c^d (\omega_1 - x_1)^2 u_{22}(x_1, \bar{x}_2 + p^*(\omega_1 - x_1)) T^G(p^*, \beta) \, dp^*$$

and IRP in Definition 2 becomes

$$\Pi_\beta = -\frac{\int_c^d (\omega_1 - x_1)^2 u_{22}(x_1, \bar{x}_2 + p^*(\omega_1 - x_1)) T^G(p^*, \beta) \, dp^*}{E_{p^*} u_2(x_1, \bar{x}_2 + p^*(\omega_1 - x_1))}. \tag{16}$$

It is immediate from (15) and (16) that IPI and IRP for price risk are zero at $x_1 = \omega_1$ and are strictly positive everywhere else.

From the properties of the indifference maps IPI has to increase as $|x_1 - \omega_1|$ increases, i.e., as $x_1$ moves away from $\omega_1$. This happens because the intersection points between the solid indifference curve and the succession of dashed indifference curves get further

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Footnote: Where $g$ is the p.d.f. associated with $G$, $T^G(t, \beta) = \int_c^d g_{\beta}(\tau, \beta) \, d\tau$ satisfies $T^G(c, \beta) = T^G(d, \beta) = 0$ and $T^G(t, \beta) \geq 0$ for all $t \in [c, d]$. 

away from the vertical line at $\omega_1$, as shown in Fig. 2. This behavior of IPI is shown in Fig. 3.

The property that IRP for price risk is zero at $x_1 = \omega_1$ guarantees that it must be increasing in some small neighborhood of $x_1 = \omega_1$ as $|x_1 - \omega_1|$ increases. We assume that IRP for price risk always increases as $|x_1 - \omega_1|$ increases. That is, it exhibits the same general behavior as the IPI function.\(^7\)

The comparative statics of increased price risk follow immediately from Theorems 2 and 3 and the behavior of IPI and IRP. Specifically, a buyer of good 1 will buy less of it as price risk increases and a seller of good 1 will sell less of it as price risk increases. Irrespective of whether the consumer is a buyer or a seller, the total effect of increased price risk is to move the consumer’s optimal choice of good 1 toward his endowment point. The same results are obtained for compensated demand. The result here reconciles seemingly contradictory results in the literature. Dardanoni (1988) found increased price risk will cause a consumer to buy less of good 1. Hanson and Menezes (1978) found that the introduction of price risk will cause a consumer to sell less of good 1. Dardanoni’s result is confined only to buyers, while Hanson-Menezes’ result is confined only to sellers. Our result, which pertains to both buyers and sellers, is consistent with these findings.

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\(^7\) The vertical distance between the two indifference curves in Fig. 2 with the same level of expected utility $\bar{v}$ but different values of the risk parameter $\beta$ measures IRP for price risk. This distance is zero at $x_1 = \omega_1$ because the consumer faces no risk at this point. It seems plausible to assume that the distance between the curves increases as $|x_1 - \omega_1|$ increases.
7. Additively separable preferences

The behaviors of IPI and IRP along an indifference curve are governed by established hypotheses about single-variable utility functions when the two-variable utility function \( u(x_1, x_2) \) is additively separable, i.e.,

\[
u(x_1, x_2) = h(x_1) + \phi(x_2).
\]

(17)

For this class of utility functions, the behavior of IPI for income risk and for price risk depend respectively on whether \( \phi \) displays aversion to (preference for) downside risk and aversion to (preference for) outside risk, while the behavior of IRP for income risk and for price risk depend respectively on whether \( \phi \) displays Ross-DARA (Ross-IARA) and increasing (decreasing) multiplicative risk aversion, a Ross-like property of preferences which has not previously appeared in the literature. From the results obtained earlier, these aspects of preferences control the uncompensated and compensated effects of increases in income risk and price risk in the case of additively separable preferences.

7.1. Preferences underlying uncompensated effects

For \( u \) given by (17), \( u_{22}(x_1, \bar{x}_2) = \phi''(\bar{x}_2) \). Hence, by Theorem 4(a), in the presence of pure income risk IPI increases (decreases) in \( x_1 \) along an indifference curve if and only if \( \phi''' > (<) 0 \). The sign of \( \phi''' \) determines the individual’s attitude to downside risk. Let \( \tilde{z} \) and \( \tilde{\varepsilon} \) be random variables such that \( E\{\tilde{\varepsilon}|\tilde{z} = z\} = 0 \) for all \( z \). The individual is downside risk averse (\( \phi''' > 0 \)) if and only if he prefers lottery \( d_1 \) to lottery \( d_2 \), where

\[
d_1 = \begin{cases} 
\frac{1}{2} w_1 + \tilde{z} + \tilde{\varepsilon} & \text{if } \phi''' > 0 \\
\frac{1}{2} w_2 + \tilde{z} & \text{if } \phi''' < 0
\end{cases} \quad d_2 = \begin{cases} 
\frac{1}{2} w_1 + \tilde{z} & \text{if } \phi''' > 0 \\
\frac{1}{2} w_2 + \tilde{z} + \tilde{\varepsilon} & \text{if } \phi''' < 0
\end{cases}
\]

and \( w_2 < w_1 \). The lotteries in this pair have the same mean and the same variance, \( d_2 \) has more downside risk than \( d_1 \) since \( d_2 \) can be obtained from \( d_1 \) by transferring dispersion \( \tilde{\varepsilon} \) from the higher wealth \( w_1 \) to the lower wealth \( w_2 \).\(^8\)

\(^8\) See Bigelow and Menezes (1995) for details on downside risk aversion and outside risk aversion defined below.
In the presence of pure price risk and for $u$ given by (17), by (15), for $x_1 > \omega_1$ IPPI increases (decreases) in $x_1$ along an indifference curve if and only if

$$O(\omega_2, t) \equiv 2\phi''(\omega_2 + t) + t\phi'''(\omega_2 + t) < (>)0$$

for $t$ in the domain of $(\omega_1 - x_1)\bar{p}$.

For $x_1 < \omega_1$, IPPI decreases (increases) in $x_1$ along an indifference curve if and only if (18) holds. The sign of $O(\omega, t)$ determines the individual’s attitude toward outside risk. Let $\tilde{z}$ and $\tilde{\varepsilon}$ be as defined above. The individual is outside risk averse ($O(\omega, t) < 0$ for all $(\omega, t)$ such that $w + t > 0$) if and only if he prefers lottery $\omega_1$ to lottery $\omega_2$, where

$$O_1 = \begin{cases} \frac{1}{2} & w_0 + \lambda_1 (\tilde{z} + \tilde{\varepsilon}) \\ \frac{1}{2} & w_0 + \lambda_2 \tilde{z} \end{cases}$$

$$O_2 = \begin{cases} \frac{1}{2} & w_0 + \lambda_1 \tilde{z} \\ \frac{1}{2} & w_0 + \lambda_2 (\tilde{z} + \tilde{\varepsilon}) \end{cases}$$

and $0 \leq \lambda_1 < \lambda_2$. $\omega_2$ can be obtained from $\omega_1$ in two steps. First, dispersion $(\lambda_1 \tilde{\varepsilon})$ is transferred from the top branch of $\omega_1$ to its bottom branch. Since $\lambda_1 < \lambda_2$, dispersion is transferred from the center of $\omega_1$ to its tails. Second, the transferred dispersion is multiplicatively increased (from $\lambda_1 \tilde{\varepsilon}$ to $\lambda_2 \tilde{\varepsilon}$). Hence, $\omega_2$ has more outside risk than $\omega_1$. Consider the following examples of utility functions

$$\phi_1(x) = \rho x - e^{-\delta x} \quad \text{and} \quad \phi_2(x) = x^\theta,$$

where $\rho \geq 0$, $\delta > 0$ and $0 < \theta < 1$. $\phi_1$ exhibits constant absolute risk aversion (CARA) for $\rho = 0$ and decreasing absolute risk aversion (DARA) for $\rho > 0$. $\phi_2$ exhibits constant relative risk aversion (CRRA). Both functions display downside risk aversion. $\phi_1$ displays outside risk aversion for $t < 2/\delta$, while $\phi_2$ uniformly displays outside risk aversion.

7.2. Preferences underlying compensated effects

For $u$ given by (17), by Theorems 3 and 4(b), in the presence of pure income risk IRP increases (decreases) in $x_1$ along an indifference curve if and only if there exists a scalar $\gamma$ such that

$$\frac{\phi''(w)}{\phi''(w)} \leq (>) \gamma \leq (>) \frac{\phi''(w)}{\phi'(w)} \text{ for all } w.$$  

\footnote{This condition puts a constraint on the size of the risk for outside risk aversion to hold.}
The unbracketed (bracketed) condition (20) is Ross-DARA (Ross-IARA). The preference meaning of Ross-DARA is that the Ross-premium, which is the amount an individual is willing to pay to avoid an increment in risk when background risk is unavoidable, decreases as wealth increases.

In the presence of pure price risk, for \( u \) given by (17), by (16), for \( x_1 > \omega_1 \) IRP increases (decreases) in \( x_1 \) along an indifference curve if and only if there exists a scalar \( \gamma \) such that

\[
2 + \frac{t \phi'''(w + t)}{\phi''(w + t)} \geq (\leq) \gamma \geq (\leq) \frac{t \phi''(w + t)}{\phi'(w + t)} \quad \text{for all } t \text{ in the support of } (\omega_1 - x_1)\hat{p}.
\]

(21)

For \( x_1 < \omega_1 \), IRP decreases (increases) in \( x_1 \) along an indifference curve if and only if (21) holds. We call the unbracketed (bracketed) condition (21) increasing (decreasing) multiplicative risk aversion. The preference meaning of increasing multiplicative risk aversion is that the premium an individual is willing to pay to avoid an increment in risk when background risk is unavoidable increases as the scale of risk increases.

For the utility functions in (19), \( \phi_1 \) displays Ross-DARA while \( \phi_2 \) does not. \( \phi_1 \) displays increasing multiplicative risk aversion as long as there exists a \( \gamma \) such that \((\rho - \gamma)/\theta < t < (2 - \gamma)/\theta\), while \( \phi_2 \) displays increasing multiplicative risk aversion for \( t \geq -(1 - \theta)w \).

In summary, for the additively separable utility function in (17), if \( \phi \) displays downside risk aversion the optimal choice of \( x_1 \) decreases as income risk increases; if \( \phi \) displays outside risk aversion the optimal choice of \( x_1 \) decreases for a buyer of good 1 and increases for a seller of good 1 as price risk increases; if \( \phi \) displays Ross-DARA the compensated choice of \( x_1 \) decreases as income risk increases; and if \( \phi \) displays increasing multiplicative risk aversion the compensated choice of \( x_1 \) decreases for a buyer of good 1 and increases for a seller of good 1 as price risk increases.

8. Concluding remarks

We conclude the paper by considering its extension to higher dimensional models and its implications for empirical demand analysis. At a formal mathematical level, a model with \( n \) period horizon can always be collapsed into one with a one-period horizon by (backward) stepwise maximization (Fama, 1970; Hadar, 1971). The complication that arises is that the resulting two-period objective function involves a value function whose preference properties are not well-known, as also noted by Aura et al. (2002). The two-good model has been widely used in the literature because of its analytical tractability and because economists believe that it captures the essence of decision making under risk.

The response of decision-makers to income and/or price risk has empirical implications for the study of consumption functions. In situations involving substantial income and/or price risk the variability of income and prices are significant variables in estimating the relationship between consumption function and other independent variables. Let

\[
C = f(y, Y, p, \sigma^2_Y, \sigma^2_p),
\]

These conditions put constraints on the size of the risk for increasing multiplicative risk aversion to hold.
where \( C \) denotes current consumption, \( y \) current income, \( Y \) expected future income, \( p \) price (or interest rate), \( \sigma^2_Y \) the variance of \( Y \), and \( \sigma^2_p \) the variance of \( p \). Our results indicate that in a linear estimation of the above equation the coefficient of \( \sigma^2_Y \) should be negative. As a result, an estimation ignoring \( \sigma^2_Y \) will lead to under-estimates of the other coefficients, as first noted by Leland (1968). Our results show that it is important to distinguish between lenders \((C < y)\) and borrowers \((C > y)\). The coefficient of \( \sigma^2_p \) should be positive for lenders and negative for borrowers. Ignoring \( \sigma^2_p \) will result in over-estimates of other variables in an equation for lenders and under-estimates of other variables in an equation for borrowers. The same considerations apply to compensated demands.

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Appendix A

Proof of Lemma 1. Since the solutions to both (3) and (10) are unique it suffices to show that either of the two demand bundles is a solution to the other problem. By the definition of the compensated demand functions and the expected expenditure function,

\[ \bar{v} \leq v(y_1, \bar{y}_2, \alpha, \beta) \quad \text{and} \quad \bar{p}y_1 + \bar{y}_2 = \bar{p}\omega_1 + \bar{\omega}_2 + e \]  

(A.1)

where we have suppressed the notation for the arguments of the functions. This means \((y_1, \bar{y}_2)\) satisfies the budget constraint for ordinary demand when the mean endowments are \((\omega_1, \bar{\omega}_2 + e)\). Therefore, since \((x_1, \bar{x}_2)\) is utility maximizing,

\[ \bar{v} \leq v(y_1, \bar{y}_2, \alpha, \beta) \leq v(x_1, \bar{x}_2, \alpha, \beta), \]  

(A.2)

so \((x_1, \bar{x}_2)\) also satisfies the utility constraint for compensated demand. Because (A.1) is an equality and \((x_1, \bar{x}_2)\) satisfies the budget constraint, \(\bar{p}y_1 + \bar{y}_2 = \bar{p}\omega_1 + \bar{\omega}_2 + e \geq \bar{p}x_1 + \bar{x}_2\). The cost of \((x_1, \bar{x}_2)\) is no greater than the cost of \((y_1, \bar{y}_2)\). In view of (A.2), \((x_1, \bar{x}_2)\) solves the compensated demand problem. \(\square\)

Proof of Theorem 2. From the definition of \( m \) in (5),

\[ m_s = \frac{v_2v_{1x} - v_1v_{2x}}{(v_2)^2} = \frac{v_{1x} - m v_{2x}}{v_2}. \]

From Definition 1 and the fact that \( \partial x_2 / \partial x_1 \big|_{x=\bar{x}} = -m \),

\[ \frac{\partial \rho_s}{\partial x_1} \bigg|_{x=\bar{x}} = -\frac{\partial v_s}{\partial x_1} \bigg|_{x=\bar{x}} = -(v_{1x} - m v_{2x}). \]

Combining these results with (7),

\[ \frac{\partial x_1}{\partial s} = \frac{m_s}{D} = \frac{1}{v_2D} \left( \frac{\partial \rho_s}{\partial x_1} \bigg|_{x=\bar{x}} \right). \]

The theorem follows immediately. \(\square\)
Proof of Theorem 3. From the definition of \( m \) in (5), calculate
\[
m_2 = \frac{v_2 v_{12} - v_1 v_{22}}{(v_2)^2} = \frac{v_{12} - m v_{22}}{v_2} \quad \text{and} \quad m_s = \frac{v_{1s} - m v_{2s}}{v_2}.
\]
From Definition 2 and the fact that \( \frac{\partial \bar{y}_2}{\partial y_1} |_{v = \bar{v}} = -m \), calculate
\[
\frac{\partial \Pi^s}{\partial y_1} |_{v = \bar{v}} = -\frac{v_2 (v_{1s} - m v_{2s} - v_2 (v_{12} - m v_{22}))}{(v_2)^2}.
\]
Combining the results of these calculations yields
\[
\frac{\partial \Pi^s}{\partial y_1} |_{v = \bar{v}} = -\frac{v_2 v_2 m_s - v_1 v_2 m_2}{(v_2)^2} = -\left( m_s \frac{v_2}{m_2} \right).
\]
Using this result with (11),
\[
\frac{\partial y_1}{\partial s} = -\frac{1}{D} \left( m_s \frac{v_2}{m_2} \right) = \frac{1}{D} \left( \frac{\partial \Pi^s}{\partial y_1} |_{v = \bar{v}} \right).
\]
The theorem follows immediately. \( \Box \)

Proof of Theorem 4. We provide a proof for part (a). The proof of part (b) is analogous and is therefore omitted. Since \( D < 0 \) and \( v_2 > 0 \), Theorem 2 implies the sign of \( \frac{\partial x_1}{\partial \alpha} \) is the opposite of the sign of \( \frac{\partial \rho}{\partial x_1} |_{v = \bar{v}} \).
\[
\frac{\partial \rho}{\partial x_1} |_{v = \bar{v}} = \int_a^b \frac{\partial [-u_{22}(x_1, \bar{x}_2 + \omega_2^* \alpha, \alpha, 0)]}{\partial x_1} \left|_{v = \bar{v}} \right) T^F(\omega_2^*, \alpha) \, d\omega_2^*. \quad (A.3)
\]
Since \( \alpha \) is an index of RS risk, \( T^F \geq 0 \) everywhere, so if \( \partial [-u_{22}]/\partial x_1 |_{v = \bar{v}} \) is uniformly negative (positive) then \( \partial \rho/\partial x_1 |_{v = \bar{v}} \) is negative (positive), so \( \partial x_1/\partial \alpha \) is positive (negative). This proves the necessary condition in part (a).

For the sufficient condition of part (a), suppose \( \partial x_1/\partial \alpha \) is positive for any price and for any distribution of endowments, but that there is some \((x_1^0, \bar{x}_2^0, t^0)\) for which
\[
\left. \frac{\partial [-u_{22}(x_1^0, \bar{x}_2^0 + t^0)]}{\partial x_1} \right|_v = \bar{v} > 0.
\]
By continuity this derivative will continue to be positive for all \( t \) in some small neighborhood of \( t^0 \). We may choose relative price \( \bar{p} \) and endowments \((\bar{\omega}_1, \bar{\omega}_2)\) such that \((\bar{x}_1^0, \bar{x}_2^0)\) is ordinary demand. Moreover we can also choose the increase in RS risk indexed by \( \alpha \) such that \( T^F \) is positive only for \( t \) in the neighborhood of \( t^0 \) in which \( \partial [-u_{22}(x_1, \bar{x}_2 + t)]/\partial x_1 |_{v = \bar{v}} > 0 \). By (A.3), this would mean \( \partial x_1/\partial \alpha \) is negative contrary to assumption. Therefore, there must be no \((x_1^0, \bar{x}_2^0, t^0)\) for which (A.4) holds. \( \Box \)

References


