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ON THE RISK–DOWNSIDE RISK TRADEOFF*

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In the last decade the literature has established the empirical importance of the tradeoff between risk and downside risk in a variety of economic settings. While the notions of risk and downside risk have been generalized in the theoretical literature, the literature has yet to provide a choice-theoretic characterization of their tradeoff. This paper provides an analytical characterization of the risk–downside risk tradeoff and shows its relevance in the analysis of optimal decisions under uncertainty, such as the precautionary savings decision.

1 Introduction

In the last decade a sizable empirical literature has focused on the distributional characteristics of financial and economic variables (e.g. Bekaert and Harvey, 1997; Bekaert *et al*., 1998; Pownall and Koedijk, 1999). This literature has established that the distributions of a wide range of financial and economic variables are highly skewed, and that omitting skewness in modeling can result in misleading conclusions.[1] In light of these findings, a number of recent studies have incorporated skewness as well as the tradeoff between skewness and variance in their analysis of economic behavior. For example, such tradeoffs have been investigated in horse race betting behavior (Golec and Tamarkin, 1998), in the purchase of state lotteries in the USA (Garrett and Sobel, 1999) and in the analysis of self-protection (Chiu, 2000). The importance of incorporating skewness is underscored by the work of Harvey and Siddique (2000) who found that systematic skewness 'commands a risk premium, on average, of 3.60 per cent per year'.

In these studies, variance and skewness are proxies for risk and downside risk. In contrast, the notions of risk and downside risk have been generalized in the theoretical literature, but this literature has yet to provide a

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[1]Pownall and Koedijk (1999) state that 'Using data on Asian equity markets, we observe that during periods of financial turmoil, deviations from the mean–variance framework become more severe, resulting in periods with additional downside risk to investors. Current risk management techniques failing to take this additional downside risk into account will underestimate the true value-at-risk with greater severity during periods of financial turmoil' (p. 853).

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choice-theoretic characterization of the tradeoff between risk and downside risk, whose importance has been emphasized in the recent empirical literature. In this paper we provide an analytical characterization of this tradeoff. To formally capture it, we present a simple class of lottery pairs that can be decomposed into a combination of lottery pairs that differ only by risk and lottery pairs that differ only by downside risk. In this class of lottery pairs, exposure to downside risk is fixed while exposure to risk increases with a parameter s (representing a mean-preserving spread). Preferences between such lotteries represent aversion to risk relative to aversion to downside risk. We show that there is a unique level of the risk-exposure parameter s for which aversion to risk just balances aversion to downside risk. It turns out that this value of s is exactly the precautionary premium, a concept that was formulated in an entirely different theoretical context and has been interpreted as the sure reduction in future income that has the same effect on current saving as does the addition of a zero-mean risk on future income.[2] Our analysis shows that the reduction in future income required to equate saving under certainty to saving under uncertainty has the distinguishing property that it balances aversion to risk and aversion to downside risk.

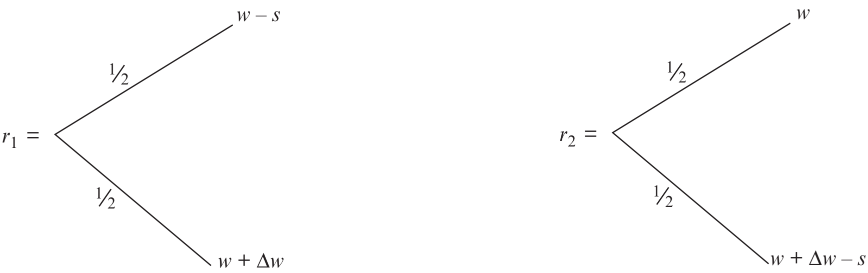
Section 2 contains our analysis of the risk–downside risk tradeoff and its relationship to the precautionary premium. Section 3 briefly reviews the literature. In it, we show how the risk–downside risk tradeoff is relevant for the analysis of precautionary saving. Section 4 concludes the paper.

2 The Risk–Downside Risk Tradeoff

In this section we describe three classes of risk pairs. The first class of risk pairs represents an increase in risk. The second class represents an increase in downside risk. The third class has the property that the strength of preference between any pair of risks in this class can be decomposed into the difference between the strength of aversion to risk pairs in the first class and the strength of aversion to downside risk pairs in the second class. We show that for any individual who is both risk averse and downside risk averse, there exists a unique pair of risks in the third class for which the individual is indifferent between the risks in this pair. That is, for this risk pair, aversion to downside risk just offsets aversion to risk. For this risk pair, the change in risk as measured by the size of a mean-preserving spread exactly offsets the change in downside risk. This mean-preserving spread is shown to be the precautionary premium defined in the literature.

Let w and Δw denote wealth and increment in wealth, respectively. Consider the class $\boldsymbol{R}$ of risk pairs (r_1, r_2), where $s \geq 0$:

[2]The precautionary premium was introduced into the literature by Kimball (1990).

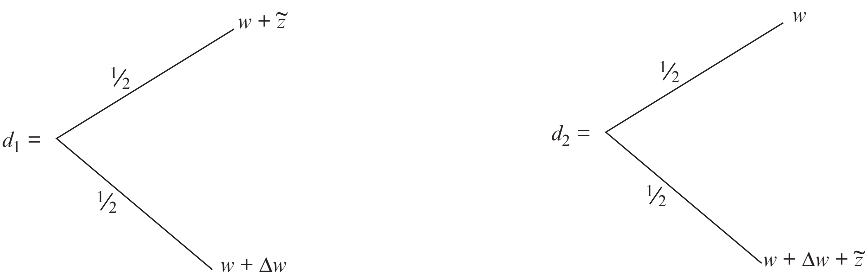


This pair has the characteristic that r_1 can be obtained from r_2 by a mean-preserving spread. Since s is the distance by which each point of the support of r_2 is moved outward to obtain r_1, s serves as a measure of the size of the spread. We call the difference between the distributions of r_1 and r_2 an *s-spread*. Since r_1 has more risk than r_2, all risk averters ($u'' < 0$) prefer r_2 to r_1. Following Friedman and Savage (1948),[3]

$$\mathrm{SR} = \mathrm{E}u(r_2) - \mathrm{E}u(r_1) \tag{1}$$

is the disutility attached to the s-spread and represents the strength of aversion to the increase in risk.[4]

Let $\tilde{z}$ be an actuarially neutral random variable (i.e. $\mathrm{E}\tilde{z} = 0$) with distribution function $F(z)$ on support $[-a, b]$, $a > 0$ and $b > 0$. Consider the class $\boldsymbol{D}$ of risk pairs (d_1, d_2):



This pair has the characteristic that d_1 can be obtained from d_2 by a mean–variance-preserving downward transfer of dispersion (actuarially

[3] $\mathrm{SR} = \frac{1}{2}\int_s^0 \int_0^{\Delta w} [-u''(w+t+x)]\mathrm{d}t\mathrm{d}x$ is positive for all risk averters ($u'' < 0$).

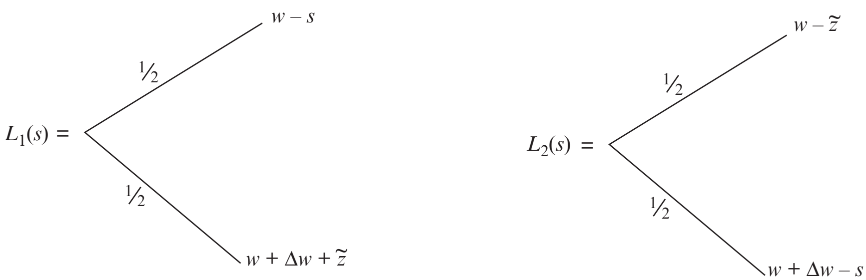
[4] $\mathrm{SR} = \mathrm{E}u(r_2) - \mathrm{E}u(r_1) = [\mathrm{E}u(r_2) - u(\bar{r})] - [\mathrm{E}u(r_1) - u(\bar{r})]$, where $\bar{r}$ is the mean value of r_1 and r_2. Following Friedman and Savage (1948), each of the bracketed terms is a utility-based measure of risk aversion since it represents the strength of an individual's preference for the sure option $\bar{r}$ over the risky alternative r_1 or r_2. It follows that the difference (i.e. SR) represents the strength of the individual's preference for r_2 over r_1.

neutral noise $\tilde{z}$ is shifted from $w + \Delta w$ to w). Hence, d_1 has more downside risk than d_2.[5] All downside risk averters ($u''' > 0$) prefer d_2 to d_1. Let[6]

$$\mathrm{SD} = \mathrm{E}u(d_2) - \mathrm{E}u(d_1) \tag{2}$$

SD is the disutility attached to the increase in downside risk and represents the strength of downside risk aversion.

Consider now the class $\boldsymbol{L}$ of lotteries ($L_1(s)$, $L_2(s)$) defined by[7]



The strength of preference for pairs in $\boldsymbol{L}$ can be decomposed into the difference between the strength of preference for pairs in $\boldsymbol{R}$ and the strength of preferences for pairs in $\boldsymbol{D}$.[8] This follows immediately from

$$\mathrm{E}u(L_2) - \mathrm{E}u(L_1) = [\mathrm{E}u(r_2) - \mathrm{E}u(r_1)] - [\mathrm{E}u(d_2) - \mathrm{E}u(d_1)] = \mathrm{SR} - \mathrm{SD} \tag{3}$$

From (3),

$$\begin{aligned} &\mathrm{E}u(L_2) - \mathrm{E}u(L_1) > (<, =)\, 0 \\ &\quad \text{iff } \mathrm{E}u(r_2) - \mathrm{E}u(r_1) > (<, =)\, \mathrm{E}u(d_2) - \mathrm{E}u(d_1) \end{aligned} \tag{4}$$

For a small s (close to zero), L_1 is preferred to L_2 and so the strength of downside risk aversion exceeds the strength of risk aversion; the reverse happens for a large s. As s increases, SR increases while SD remains unchanged.[9] That

[5] Menezes *et al.* (1980) provided the characterization of downside risk in terms of integral conditions.

[6] $\mathrm{SD} = \frac{1}{2}\int_{-a}^{b}\int_{0}^{\Delta w}\int_{0}^{z}(z-y)u'''(w+t+y)\,\mathrm{d}y\,\mathrm{d}t\,\mathrm{d}F(z)$ is positive for downside risk averters ($u''' > 0$).

[7] For any $s \geq 0$, the lotteries L_1 and L_2 have the same mean, and the variance of L_1 is greater than or equal to that of L_2 since $\mathrm{var}(L_1) - \mathrm{var}(L_2) = (\Delta w)s \geq 0$.

[8] Let F^{α} denote the distribution function for any lottery. It can be shown that

$$F^{L_1} - F^{L_2} = (F^{r_1} - F^{r_2}) + (F^{d_2} - F^{d_1})$$

It follows that the comparative structure of L_1 and L_2 is a mixture of risk and downside risk.

[9] $\dfrac{\partial \mathrm{SR}}{\partial s} = \dfrac{1}{2}\int_{0}^{\Delta w}[-u''(w+t-s)\mathrm{d}t > 0]$. Since $\dfrac{\partial^2 \mathrm{SR}}{\partial s^2} = \dfrac{1}{2}\int_{0}^{\Delta w}u'''(w+t-s)\mathrm{d}t > 0$ SR is convex in s.

is, the strength of risk aversion increases as s increases and the strength of aversion to downside risk remains unchanged as s increases. This leads to the following result.

Proposition 1: If $u'' < 0$ and $u''' > 0$ then there exists a unique value of s for which $\mathrm{E}u(L_1) = \mathrm{E}u(L_2)$, or equivalently the strength of risk aversion is equal to the strength of downside risk aversion (SR = SD).

Let s^* be the value of s for which SR = SD. We now show how s^* is related to the precautionary premium γ, which was introduced by Kimball (1990) and is defined by the equation $u'(w - \gamma) = \mathrm{E}u'(w + \tilde{z})$. Rearranging the equation SR = SD gives

$$u(w + \Delta w - s^*) - u(w - s^*) = \mathrm{E}u(w + \Delta w - \tilde{z}) - \mathrm{E}u(w + \tilde{z}) \qquad (5)$$

For an infinitesimal increase in wealth ($\Delta w \to 0$), (5) becomes

$$u'(w - s^*) = \mathrm{E}u'(w + \tilde{z}) \qquad (6)$$

Hence, s^* is equal to the precautionary premium γ. Proposition 1 provides a new interpretation for the precautionary premium. That is, the precautionary premium is the change in risk as measured by the size of a mean-preserving spread that just offsets a given change in downside risk.

Figure 1 graphically depicts the precautionary premium. In it, the curves SR and SD are drawn from a fixed level of wealth w. The precautionary premium γ is the value of s where the curves SR and SD intersect. For $s < \gamma$,

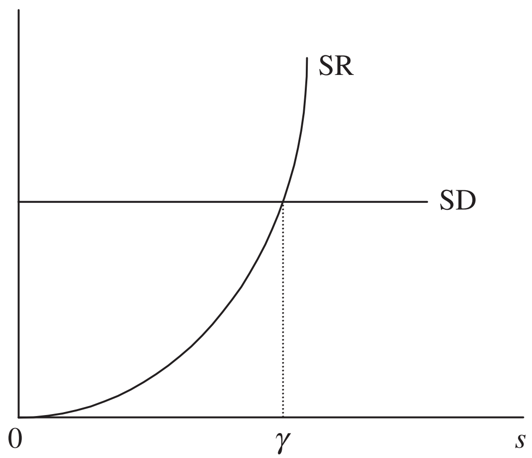


Fig. 1

SR lies below SD, indicating that the strength of downside risk aversion exceeds that of risk aversion. The reverse is true for $s > \gamma$.[10]

3 Literature

Precautionary saving refers to saving that arises from uncertainty about future income. Irving Fisher (1930) attributed such saving to the need 'to lay up for a rainy day', arguing that 'the greater the risk of rainy days in the future, the greater is the impulse to provide for them at the expense of the present' (pp. 77–78). Fisher's insight was formalized by contributions by Leland (1968) and Sandmo (1970), who analyzed precautionary saving in a two-period expected utility framework. They showed that precautionary saving is positive (negative) according as the individual is a downside risk averter (preferrer).

Kimball (1990) extended the analysis of precautionary saving by considering the intensity of the precautionary saving motive, as measured by the precautionary premium. He defined this premium as the sure reduction in future income that has the same effect on saving as the addition of a zero-mean risk to future income. We can illustrate precautionary saving and its relationship to the precautionary premium in a simple two-period model. This enables us to show how the risk–downside risk tradeoff underlies the intensity of the precautionary saving motive.

Let k^c be the optimal saving under certainty:

$$\max_k v(w_1 - k) + u(w_2 + k) \tag{7}$$

where w_1 and w_2 are income in period 1 and period 2, respectively, v is the period-one utility function. Let k^u be the optimal saving under income risk when period-two income is $w_2 + \tilde{z}$:

$$\max_k v(w_1 - k) + \mathrm{E}u(w_2 + \tilde{z} + k) \tag{8}$$

Finally, let $k^*(R)$ denote the optimal saving when second period income is $w_2 - R$:

$$\max_k v(w_1 - k) + u(w_2 - R + k) \tag{9}$$

where R is the reduction in period-two income. The precautionary saving is given by $k^u - k^c$. It is positive under downside risk aversion ($u''' > 0$). The precautionary premium γ is the value of R for which $k^* = k^u$. From the first-order conditions for (8) and (9), this happens when $u'(w - \gamma) = \mathrm{E}u'(w + \tilde{z})$, where $w = w_2 + k^*$.

Figure 2 illustrates precautionary saving and the precautionary premium in terms of the relationship between the solutions to the problems (7)–(9). S^c,

[10]For $s < \gamma$, $u'(w - s) < \mathrm{E}u'(w + \tilde{z})$; for $s > \gamma$, $u'(w - s) > \mathrm{E}u'(w + \tilde{z})$.

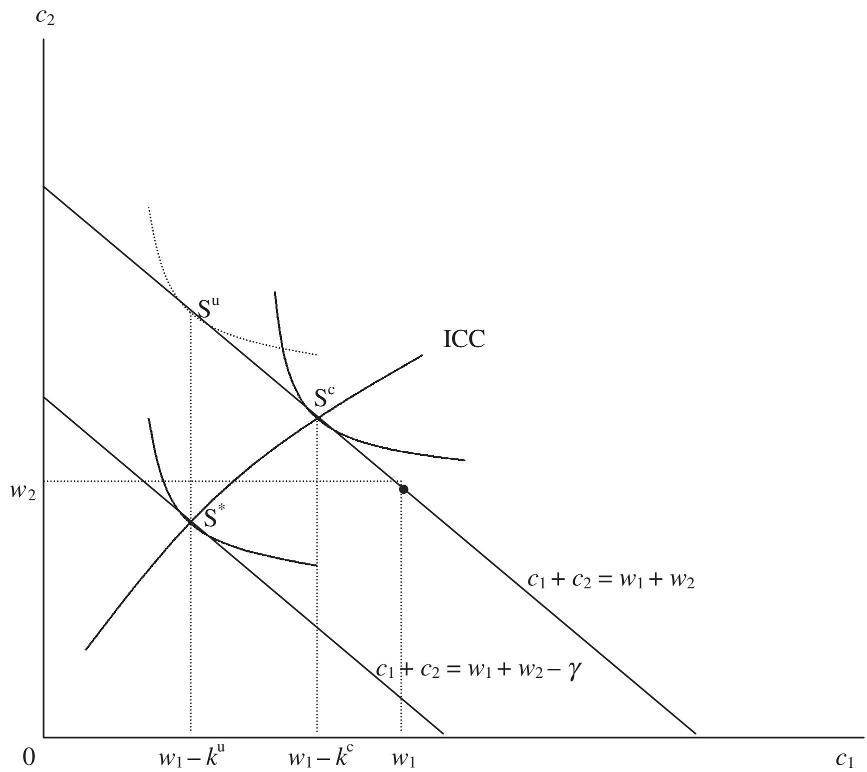


FIG. 2

S^u and S* are the optimal points for (7), (8) and (9), respectively, where S* corresponds to $R = \gamma$.[11] In the diagram, precautionary saving is the horizontal distance between S^c and S^u while the precautionary premium γ is the vertical distance between S^u and S*.

We now show the sense in which the risk–downside risk tradeoff underlies the intensity of the precautionary saving motive. In Fig. 2, the curve connecting S^c and S* is the income–consumption curve (ICC) under certainty resulting from changes in period-two income. Suppose $R < \gamma$. Then the optimal solution to (9) is at some point on ICC that is above S*. Hence, saving under income risk (k^u) is greater than saving (k^*) under certainty with second period income $w_2 - R$. We next show that this is equivalent to the individual's

[11]The introduction of a zero-mean risk $\tilde{z}$ causes indifference curves to shift upward due to risk aversion and to rotate counterclockwise due to downside risk aversion. Hence, S^u lies to the left of S^c on the budget line: $c_1 + c_2 = w_1 + w_2$. Note that for the uncertainty case it is the expected budget line. See Menezes and Auten (1978).

preference between two lotteries that involve a tradeoff between risk and downside risk and for which the individual's aversion to risk is dominated by aversion to downside risk.

By the first-order conditions for (8) and (9) and with $k^{u} > k^*$,

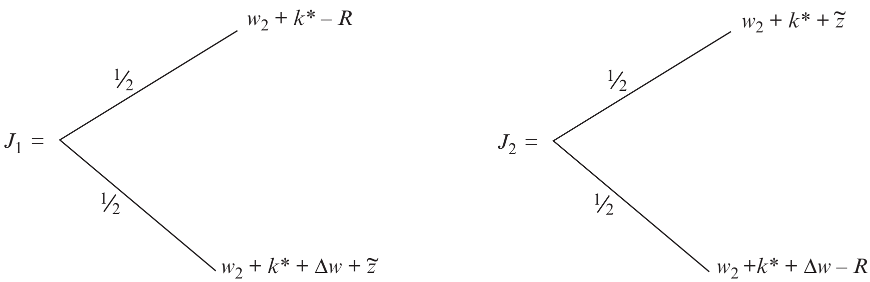
$$-v'(w_1 - k^*) + \mathrm{E}u'(w_2 + k^* + \tilde{z}) > 0$$

$$-v'(w_1 - k^*) + u'(w_2 - R + k^*) = 0$$

Hence,

$$\mathrm{E}u'(w_2 + k^* + \tilde{z}) > u'(w_2 - R + k^*)$$

This holds if and only if J_1 is preferred to J_2 for small $\Delta w > 0$, where



Matching w with $w_2 + k^*$ and s with R, J_1 and J_2 respectively become L_1 and L_2 defined in Section 2. Hence, from the analysis in Section 2, aversion to downside risk dominates aversion to risk. The intuition underlying this is that the exposure to risk as determined by the size of the spread (R) is lower than the exposure to risk when the size of the spread is γ. Since downside risk is unaffected by the size of the spread, exposure to downside risk dominates that of risk. The reverse happens when $R > \gamma$. Finally, when $R = \gamma$, aversion to risk just offsets aversion to downside risk.

4 Conclusion

The empirical literature has established that risk considerations alone cannot account for a variety of economic phenomena, and that downside risk and the tradeoff between risk and downside risk must be included in order to explain observed behavior. This paper analytically characterizes this tradeoff in a simple setting. Our analysis provides an intuitive alternative interpretation of the precautionary premium and is relevant for the analysis of optimal decisions under uncertainty.

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