INVERSE ESTIMATION OF SURFACE TEMPERATURE INDUCED BY A MOVING HEAT SOURCE IN A 3-D OBJECT BASED ON BACK SURFACE TEMPERATURE WITH RANDOM MEASUREMENT ERRORS

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Temperatures on inaccessible surfaces can be estimated by solving an inverse heat conduction problem (IHCP) based on the measured temperature on accessible surfaces. In this article, the transient temperatures on the front (heated) surface of a three-dimensional (3-D) object is recovered using the conjugate gradient method (CGM) based on temperatures measured on the back surface (opposite to the heated surface). The simulated measurement data are generated from the exact solution obtained by solving the direct problem, in which the front surface of the object is subjected to a moving heat source having an elliptic spot with a Gaussian profile. Random errors are artificially imposed on the back surface data. It is shown that the front-surface temperatures of the 3-D object can be well recovered using the algorithm presented here. Studies are also carried out to test the effect of large measurement errors on the accuracy of the inverse solution.

1. INTRODUCTION

High-energy laser (HEL) weapons can remotely deliver high power at the speed of light onto a military target. It is critical to know the transient temperature in the target in order to accurately assess the resulting thermomechanical response. However, conventional temperature sensors cannot be used to directly measure the surface temperature, since the sensors can be easily destroyed or interfere with the laser beam. Similar inverse problems can be found during re-entry of a space vehicle into the atmosphere as well as in high-power laser manufacturing processes [1]. For these
In certain circumstances, the heated surface temperature must be determined indirectly by solving an inverse heat conduction problem (IHCP) based on the transient temperature measured on the back surface. Over the past several decades, various methods including analytical and numerical approaches have been proposed to solve the IHCPs [e.g., 2–4]. Since most of the analytical solutions are limited to the IHCPs with constant material properties and simple geometry [5–9], researchers have to resort to numerical approaches when multidimensional nonlinear inverse problems are involved [10–15]. The majority of the numerical methods restate the inverse problem as a least-squares minimization problem over the whole-time domain or in sequential time intervals. To improve the stability of the inverse solution, a smoothing term, which is referred to as regularization parameter, is usually added to the least squares norm [16]. However, the optimal value of this parameter is often difficult to acquire. For this reason, an iterative regularization approach has been developed in which the regularization procedure is performed through the iterative processes and thus, the determination of the optimal regularization parameter is not required [17].

Among various iterative regularization techniques, the conjugate gradient method (CGM) has been extensively applied to solve IHCPs due to its excellent self-adjusting convergence property. For example, Huang and Wang [18] used the CGM to estimate surface heat fluxes for a three-dimensional inverse heat conduction problem. Loulou and Scott [19] employed the CGM to retrieve the time-dependent

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<thead>
<tr>
<th>NOMENCLATURE</th>
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<tbody>
<tr>
<td>$c_p$</td>
<td>specific heat, J/(kg · K)</td>
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<td>$d^k$</td>
<td>direction of descent at iteration $k$, which is sometimes expressed in vector form $d^k$</td>
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<td>$h$</td>
<td>convection heat transfer coefficient, W/(m² · K)</td>
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<td>$k$</td>
<td>thermal conductivity, W/(m · K)</td>
</tr>
<tr>
<td>$L, M, N$</td>
<td>object lengths in $x$, $y$, and $z$ directions, m</td>
</tr>
<tr>
<td>$q$</td>
<td>heat flux, W/m²</td>
</tr>
<tr>
<td>$q_0$</td>
<td>moving heat source on the front surface, W/m²</td>
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<tr>
<td>$q_1$</td>
<td>observed heat flux on front surface which is sometimes expressed in vector form $q_1$, W/m²</td>
</tr>
<tr>
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<td>objective function</td>
</tr>
<tr>
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<td>time, s</td>
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<tr>
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<td>temperature, K</td>
</tr>
<tr>
<td>$T_0$</td>
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</tr>
<tr>
<td>$T_\infty$</td>
<td>ambient temperature, K</td>
</tr>
<tr>
<td>$T_1$</td>
<td>front surface temperature, K</td>
</tr>
<tr>
<td>$x, y, z$</td>
<td>spatial coordinate variables, m</td>
</tr>
<tr>
<td>$Y_{T\text{exact}}$</td>
<td>exact temperature on back surface obtained by numerical simulations, K</td>
</tr>
<tr>
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<td>$\alpha$</td>
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<tr>
<td>$\beta^k$</td>
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</tr>
<tr>
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<td>temperature variation, K</td>
</tr>
<tr>
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<tr>
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Superscript
blood perfusion rate in biological tissues based on heat flux measurements. Abboudi and Artioukhine [20] utilized the CGM to recover the surface heat flux evolutions from the transient temperature histories taken with several sensors inside a two-dimensional specimen. Recently, the authors applied the CGM to reconstruct the front-surface heating condition for a 3-D object with temperature-dependent thermophysical properties based on the back-surface heat flux and temperature measurement data [21].

However, most of the previous CGM schemes dealt with the recovery of fixed surface heating condition. In real field conditions, the surface heat source may be a moving one with irregular spot shape. In the past, a few studies were related to the recovery of moving heat sources but the moving heat sources were inside the object instead of surface [22]. To the best of the authors’ knowledge, no work has been done in the estimation of the moving heat flux on the surface of a 3-D object based on temperature measurements on the opposite surface. In this study, a CGM approach is formulated to reconstruct the dynamic heating condition on the front surface of a 3-D object based on the back-surface temperature measurements with random measurement errors. The effect of large random errors on the inverse solution will also be examined.

2. MODEL DESCRIPTION

To illustrate the methodology of the inverse heat transfer algorithm employed in this study, a three-dimensional object shown in Figure 1 is considered. Initially, the object is under a uniform temperature \( T_0 \) and is then subjected to a moving heat source \( q_0(y, z, t) \) with elliptic spot shape on the front surface from \( t = 0^+ \). The purpose of the inverse algorithm is to recover the temperature distribution \( T_1(y, z, t) \) on the front surface based on the temperature measurement data \( Y_{TL}(y, z, t) \) on the back surface \( (x = L) \). In this study, the front-surface temperature \( T_1(y, z, t) \) is estimated in an indirect way, i.e., first the heat flux \( q_1(y, z, t) \) on the front surface is recovered using the inverse algorithm, and then the temperature \( T_1(y, z, t) \) on the front surface is obtained as a by-product.

In the CGM approach, the inverse heat conduction problem is transformed into several sub-problems, i.e., the direct, adjoint, and sensitivity problems, which will be presented as follows [4].

![Figure 1. Physical model.](image-url)
2.1. Direct Problem

The direct problem can be expressed as follows.

\[
\rho c_p(T) \frac{\partial T}{\partial t} = \frac{\partial}{\partial x} \left( k(T) \frac{\partial T}{\partial x} \right) + \frac{\partial}{\partial y} \left( k(T) \frac{\partial T}{\partial y} \right) + \frac{\partial}{\partial z} \left( k(T) \frac{\partial T}{\partial z} \right)
\]

for \( 0 < x < L, \ 0 < y < M, \ 0 < z < N, \ t > 0 \) \hspace{1cm} (1)

\[ T = T_0 \text{ for } 0 \leq x \leq L, \ 0 \leq y \leq M, \ 0 \leq z \leq N, \ t = 0 \] \hspace{1cm} (2)

\[-k(T) \frac{\partial T}{\partial x} = q_1(y, z, t) \text{ for } x = 0, \ t > 0 \] \hspace{1cm} (3)

\[-k(T) \frac{\partial T}{\partial x} = 0 \text{ for } x = L, \ t > 0 \] \hspace{1cm} (4)

\[-k(T) \frac{\partial T}{\partial y} = 0 \text{ for } y = 0, \ M; \ t > 0 \] \hspace{1cm} (5)

\[-k(T) \frac{\partial T}{\partial z} = 0 \text{ for } z = 0, \ N; \ t > 0 \] \hspace{1cm} (6)

In the direct problem described above, the front-surface heat flux \( q_1(y, z, t) \) is given. The objective is to determine the transient temperature distribution on the back surface. As can be seen in Eq. (1), the thermophysical properties are considered temperature-dependent, which makes the IHCP a nonlinear one.

2.2. Inverse Problem

For the inverse problem, the back surface temperature \( Y_{TL}(y, z, t) \) is given, and the heat flux on the front surface \((x = 0), q_1(y, z, t) \) is unknown and needs to be recovered along with the front surface temperature.

If the measurement data are dense in time, they can be approximated as continuous. For this case, the inverse solution can be obtained by minimizing the following ordinary least squares norm.

\[
S[q_1] = \sum_{i=1}^{m} \int_{0}^{t_f} \left( Y_{TL}(y_i, z_i, t) - T[L, y_i, z_i, t; q_i] \right)^2 dt
\] \hspace{1cm} (7)

where \( T[L, y_i, z_i, t; q_i] \) is the computed temperature at measuring point \( i \) on the back surface.

2.3. Conjugate Gradient Method for Minimization

The iterative process to estimate the front surface heat flux \( q_1(y, z, t) \) at iteration \( k + 1 \) is advanced by

\[
q_1^{k+1}(y, z, t) = q_1^k(y, z, t) - \beta^k d^k(y, z, t)
\] \hspace{1cm} (8)
where $\beta^k$ is the search step size from iteration $k$ to $k + 1$, which will be addressed in the next section, and $d^k(y, z, t)$ is the direction of descent (i.e., search direction) given by

$$d^k(y, z, t) = \nabla S[q^k_t] + \gamma^k d^{k-1}(y, z, t)$$

which is a conjugation of the gradient direction $\nabla S[q^k_t]$ at iteration $k$, and the direction of descent $d^{k-1}(y, z, t)$ at iteration $k - 1$. The conjugate coefficient $\gamma^k$ is determined by

$$\gamma^k = \frac{\sum_{i=1}^{m} \int_0^{\tau_f} \nabla S[q^k_t] \cdot \{ \nabla S[q^k_t] - \nabla S[q^{k-1}_t] \} dt}{\sum_{i=1}^{m} \int_0^{\tau_f} (\nabla S[q^{k-1}_t])^2 dt}$$

with $\gamma^0 = 0$. To perform the iterations in Eq. (8), the step size $\beta^k$ and the gradient of the objective functional $\nabla S[q^k_t]$ need to be determined. To do so, a sensitivity problem and an adjoint problem are constructed in the following.

### 2.4. Sensitivity Problem and Search Step Size

The limiting approach is employed to the governing equation, boundary, and initial conditions of the direct problem [4]. After some manipulations, the sensitivity problem is described below.

$$\frac{\partial (pcp\Delta T)}{\partial t} = \frac{\partial^2 (k\Delta T)}{\partial x^2} + \frac{\partial^2 (k\Delta T)}{\partial y^2} + \frac{\partial^2 (k\Delta T)}{\partial z^2}$$

for $0 < x < L, 0 < y < M, 0 < z < N, t > 0$

$$\Delta T(x, y, z, 0) = 0 \quad \text{for} \quad 0 \leq x \leq L, 0 \leq y \leq M, 0 \leq z \leq N, t = 0$$

$$-\frac{\partial (k\Delta T)}{\partial x} = \Delta q_1(y, z, t) \quad \text{for} \quad x = 0, t > 0$$

$$-\frac{\partial (k\Delta T)}{\partial x} = 0 \quad \text{for} \quad x = L, t > 0$$

$$-\frac{\partial (k\Delta T)}{\partial y} = 0 \quad \text{for} \quad y = 0, M; \quad t > 0$$

$$-\frac{\partial (k\Delta T)}{\partial z} = 0 \quad \text{for} \quad z = 0, N; \quad t > 0$$

The sensitivity problem described by the above equations is used to determine the temperature variation $\Delta T(x, y, z, t)$ caused by the small perturbation in heat flux $\Delta q_1(y, z, t)$.

The search step size $\beta^k$ can be determined by minimizing the objective function with respect to $\beta^k$.

$$\beta^k = \frac{\sum_{i=1}^{m} \int_0^{\tau_f} \{ T[L, y, z, t; q^k_1] - Y_{TL}(y, z, t) \} \cdot \Delta T[L, y, z, t; d^k] \cdot dt}{\sum_{i=1}^{m} \int_0^{\tau_f} (\Delta T[L, y, z, t; d^k])^2 \cdot dt}$$
where the temperature variation $\Delta T[L, y_i, z_i; t; d^k]$ is obtained by solving the sensitivity problem (Eqs. (11)–(16)) by letting $\Delta q_1(y, z, t) = d^k(y, z, t)$.

### 2.5. Adjoint Problem and Gradient Equation

The adjoint problem can also be obtained from the limiting approach.

The gradient of the least squares norm is as follows.

$$\nabla S[q_1(y, z, t')] = \lambda(0, y, z, t)$$

The discrepancy principle [4] is used as the stopping criterion for the iterative process.

### 3. GENERATION OF SIMULATED MEASUREMENT DATA

The temperature measurement data on the back surface are generated by adding a random error term to the exact solution obtained by solving a direct problem, described by Eqs. (1), (2), and (4)–(6) with the front-surface boundary condition given by

$$-k(T) \frac{\partial T}{\partial x} = q_i(y, z, t) = q''(y, z, t) = h(T - T_\infty) - \varepsilon \sigma(T^4 - T_\infty^4)$$

where $q''(y, z, t)$ is a moving heat source.

If the exact solution of back-surface temperature of the direct problem (Eqs. (1), (2), (4)–(6), and (24)) is denoted by $Y_{T\text{lexac}}(y, z, t)$, the simulated measurement temperature can be obtained as follows.

$$Y_{TL}(y, z, t) = Y_{T\text{lexac}}(y, z, t) + \zeta \varphi$$
where \( \varphi \) is the standard deviation of the measurement noise, and \( \zeta \) is a random normal variable with zero mean and unitary standard deviation. The measurement data obtained by Eq. (25) contain random errors that have a normal distribution with standard deviation equal to \( \varphi \).

### 4. RESULTS AND DISCUSSION

#### 4.1. Simulation Parameters and Sensor Arrangement

Take a thin stainless steel slab as an illustrative example. The dimension of a 3-D object considered is \( L \times M \times N = 3 \text{mm} \times 120 \text{mm} \times 120 \text{mm} \). Initially, the object is at uniform temperature 300 K. The density of the 3-D object is considered to be constant and uniform: \( \rho = 7570 \text{kg/m}^3 \). However, the thermal conductivity and specific heat are temperature-dependent as follows.

\[
\begin{align*}
\kappa(T) &= 12.45 + (1.140 \times 10^{-2} + 2.517 \times 10^{-6} T)T \\
\rho_c(T) &= 470 + 0.175 \cdot T
\end{align*}
\]

Other simulation parameters are: \( T_\infty = 300 \text{K} \), \( h = 5 \text{W/(m}^2\cdot \text{K)} \), and \( \varepsilon = 0.92 \).

The heat flux on the front surface is assumed to be in the following form.

\[
q''(y, z, t) = \alpha \cdot q''_0 \cdot \exp \left\{ -0.5 \left[ (y - y_c) \cos \omega t + \sin \omega t (z - z_c) \right]^2 / S_y^2 \right\} - 0.5 \left[ -(y - y_c) \sin \omega t + (z - z_c) \cos \omega t \right]^2 / S_z^2
\]

where \( \alpha \) is the surface absorptivity, \( \alpha = 0.2 \); \( q''_0 \) is the incident heat flux at the spot center, \( q''_0 = 2500 \text{W/cm}^2 \); \( (0, y_c, z_c) \) are the coordinates of the heating flux spot, \( y_c = 0.06 + 0.02 \cos \omega t \text{ (m)} \), \( z_c = 0.06 + 0.02 \sin \omega t \text{ (m)} \); \( S_y, S_z \) are parameters for defining heating flux spot shape; and \( \omega \) is the angular frequency. The overall effect of the front-surface heating flux for \( S_y = 18 \text{mm} \), \( S_z = 7 \text{mm} \), and \( \omega = \pi/4 \) is shown in Figure 2. As seen in Figure 2, the front-surface heating flux is a moving one with an elliptic spot shape.

The back-surface temperatures are assumed to be measured by a square array of 24 \times 24 sensors mounted on the back surface. The sensor array is centered on the back surface, and the sensors are evenly spaced, as shown in Figure 3. For simplicity of illustration, only a 10 \times 10 sensor array is plotted in Figure 3. The solid lines represent the interfaces of control volumes. The black circular dots represent the locations of the temperature sensors, which are exactly located at the centers of the control volumes.

Corresponding to this sensor arrangement, a finite difference mesh \( 9 \times 26 \times 26 \) is established to perform the numerical simulation based on the Practice B discretization method [23]. This is because two boundary nodes in each direction are needed to be included in the grid system in Practice B. As an example, two boundary nodes are shown in Figure 3 (the two open circles). These two boundary nodes plus the 24 interior nodes constitute in total 26 nodes in each coordinate direction. The standard deviation of the random errors in the measurement data is set to be 2 K if not otherwise specified. The sensors measure the back-surface temperatures at the sampling
interval 0.005 s. The time step in the finite difference simulations is set to be the same as the data sampling interval. Each simulation case is conducted from 0 through 8 s.

4.2. Recovery of Temperatures on the Front Surface

Figure 4 shows the contour comparison between the exact and recovered temperature distributions on the front surface from 0 to 7 s. The data at 8 s are discarded due to the fact that the accuracy of the inverse solution is seriously degraded around the ending time of the simulation when the CGM approach is used. In the legends of Figure 4, the data

Figure 3. Temperature sensor array on the back surface.
400 stands for a data range between 300–400 K, and the data 600 stands for a data range between 400–600 K. Other data can be explained in the similar way. As seen in Figure 4, the recovered temperatures on the front surface agree well with the exact temperatures.
Figure 5a presents a comparison between the inversed temperature and the exact solution at the front surface center. Figure 5b shows the difference between the two curves in Figure 5a (inverse solution minus the exact solution). It can be seen in Figure 5b that the difference between the inversed temperatures and the exact
solutions are within the range $-10 \text{ K} - 10 \text{ K}$. This data range is very small compared to the temperature rise (about 1400 K); therefore, the two curves are quite close in Figure 5a.

Figure 6 plots the maximum and root mean square (RMS) errors between the exact and recovered front-surface temperatures from 0 to 7 s. The maximum and RMS errors can reflect the overall quality of the inverse algorithm since they are computed over the entire front surface. It appears from Fig. 6 that the maximum and RMS errors are around 9 K and 3 K, respectively, for the case considered.

Figure 7 presents the 2-D spatial distributions of the errors between the exact and recovered temperatures on the front surface from 0 to 7 s. It is observed that the errors are between $-10 \text{ K}$ and $12 \text{ K}$. The distributions of the errors are quite random in nature on the front surface.
Figure 7. Errors (K) between the exact and recovered front-surface temperatures from 0 to 7 s: (a) $t = 1$ s; (b) $t = 2$ s; (c) $t = 3$ s; (d) $t = 4$ s; (e) $t = 5$ s; (f) $t = 6$ s; (g) $t = 7$ s (color figure available online).
In reality, the measurement data may be subjected to substantial random noise due to poor sensor mounting and unfavorable measurement environment. Figure 8 examines the effect of large random errors in the measurement data on the accuracy of the inversed solutions. In Fig. 8, three cases with different standard deviations of random errors are investigated. As can be seen, as the standard deviations ($\sigma$) increase from 2 K to 15 K, the maximum errors between the inversed temperatures and the exact solutions increase from about 9 K to about 35 K.

In the above analysis, the total number of temperature sensors is $24 \times 24$. Figure 9 presents the inversed temperature distributions (at $t = 5$ s) when the sensor numbers are $20 \times 20$ and $15 \times 15$, respectively. Comparing Fig. 9 with Fig. 4j, it can be seen that as the sensor number is reduced, the resolution of the 2-D temperature
distributions become worse though the overall shape of the heating spot can still be estimated. The determination of the sensor number depends on the compromise between accuracy requirement and sensor manufacturing.

Figure 10 shows the effect of the angular frequency of the moving heating flux on the accuracy of the inversed solutions. It is observed that when the angular frequency of the moving heating flux is $\pi/6$ or $\pi/4$, the maximum error in the recovered temperatures is below 15 K. But when the angular frequency is increased to $\pi/2$ per second, the maximum error rapidly rises to over 30 K. More extensive research on the effect of the speed of the moving heating flux on the inverse solutions is in progress in our group for different materials and geometrical dimensions.

5. CONCLUSION

A nonlinear conjugate gradient method (CGM) algorithm is formulated to recover the heat flux and temperature on the front (heated) surface of a 3-D object with temperature-dependent thermophysical properties based on the temperature measurements on the back surface (opposite to the heated surface). The dynamic temperatures on the front surface are caused by a moving heat flux with an elliptic spot impinged on the same surface. The inverse heat conduction simulation shows that the transient temperatures on the front surface can be well recovered using the inverse algorithm formulated in this study. Efforts are also made to examine the effect of large random errors in the measurement data on the accuracy of the inversed solutions. It is found that for steel with thickness of 3 mm considered in this study, when the standard deviation of the random errors in the measurement data is less than 2 K, the maximum error in the recovered temperatures on the front surface can be controlled within 10 K. The influences of sensor number and speed of moving heating flux are also examined. It is shown that as the sensor number is reduced, the resolution of the 2-D temperature distributions become worse though the overall
shape of the heating spot can still be reasonably estimated. When the angular frequency of the moving heating flux is less than $\frac{\pi}{4}$, the maximum error in the recovered temperatures will not exceed 15 K. But when the angular frequency is increased to $\frac{\pi}{2}$, the maximum error rapidly rises up to 30 K.

REFERENCES