An axisymmetric dual-phase-lag bioheat model for laser heating of living tissues

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A two-dimensional (2D) axisymmetric dual-phase-lag (DPL) model is proposed to describe heat transfer in living biological tissues with nonhomogeneous inner structures. The DPL constitutive equation is incorporated, for the first time, into the 2D axisymmetric bioheat transfer model in living tissues, and corresponding numerical approach is developed. Two heating schemes (surface heating and body heating) and two beam profiles (flat beam and Gaussian beam) are examined in detail. The numerical results indicate that the DPL bioheat conduction model describes different thermal responses from the thermal-wave and Pennes' bioheat conduction models, depending on the values of the two lagging times. For a local heating with the heated spot smaller than the tissue bulk, the variations of the non-uniform distributions of temperature suggest that the multi-dimensional effects of thermal wave and diffusion not be negligible. It is also found that due to the presence of blood perfusion in living tissues, the present DPL bioheat conduction model reduces to the Pennes' bioheat model only when \( \tau_q = \tau_T = 0 \).

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1. Introduction

For the past several decades, temperature predictions for biological bodies have attracted great attentions due to its significance in clinical, basic, aeronautic, environmental sciences, etc. Most analyses are based on the well-known Pennes' bioheat equation [1]:

\[
\rho c \frac{\partial T}{\partial t} = \nabla \cdot (k \nabla T) + w_b \rho_b c_b (T_b - T) + Q_m + Q
\]

The heat conduction term in the Pennes' bioheat equation (1) is based on the classical Fourier's law that implies an infinite thermal propagation speed. To remedy this physically unreasonable deficiency, the Pennes' bioheat equation has been modified by including the thermal wave effects [2–8]:

\[
\begin{align*}
\rho c \frac{\partial T}{\partial t} &= \nabla \cdot (k \nabla T) + w_b \rho_b c_b (T_b - T) + Q_m + Q \\
q(r, t) + \tau_T \frac{\partial q(r, t)}{\partial t} &= -k \nabla T(r, t)
\end{align*}
\]

The theoretical value of thermal relaxation time \( \tau \) in biological systems was predicted to be 20–30 s [3], and a value of approximately 16 s was experimentally observed [4]. Although the non-Fourier thermal wave model [2–8] can solve the paradox of infinite thermal propagation speed, it cannot capture the microscale response in space [9] and may introduce some unusual behaviors and physically impossible solutions such as negative thermal energies [10,11].

More recently, Antaki [12] has employed the dual-phase-lag (DPL) heat conduction model, which is based on the well-known two phase lags concept [13], to interpret the non-Fourier heat conduction phenomena in processed meats:

\[
q(r, t) + \tau_T \frac{\partial q(r, t)}{\partial t} = -k \left\{ \nabla T(r, t) + \tau_T \frac{\partial \nabla T(r, t)}{\partial t} \right\}
\]

where \( \tau_T \) is the phase lag time for temperature gradient, and \( \tau_q \) is the phase lag time for heat flux. He showed that the DPL heat conduction model correlates better with experimental data, compared to the hyperbolic thermal wave and Fourier heat conduction models.

Eq. (3) shows that the temperature gradient established across a tissue volume located at a position \( r \) at time \( t + \tau_T \) results in a heat flux to flow at a different instant of time \( t + \tau_q \). For biological tissues, \( \tau_T \) represents a measure of the time delay in conduction that occurs along microscopic paths (e.g., within meat particles) and \( \tau_q \) is a measure of the time delay in conduction (e.g., contact resistance between different tissue constituents) [12]. Eq. (3) reduces to Eq. (2) when \( \tau_T = 0 \) and to the classical Fourier law by further letting \( \tau_q = 0 \).

Currently, there exists a lot of controversy in the literature about whether or not DPL conduction and, more generally, any non-Fourier conduction is important for biological tissues. Specifically, there is limited experimental evidence for this phenomenon, and some of the non-Fourier evidence has been called into question repeatedly. An excellent review on this topic can be found in [14].

Although the validity of various non-Fourier models is debatable, no ultimate conclusion has been drawn at present due to the...
Nomenclature

\( c_b \) specific heat of blood \( J/(kg{\cdot}K) \)
\( c \) specific heat of tissues \( J/(kg{\cdot}K) \)
\( k \) thermal conductivity of tissue \( W/(m{\cdot}K) \)
\( L_x \) total number of grid points in \( x \)-direction
\( M_r \) total number of grid points in \( r \)-direction
\( q_p \) incident heat flux \( W/m^2 \)
\( q_{p,\text{max}} \) peak value at the spot center for a Gaussian-type heating flux case \( W/m^2 \)
\( q_r \) heat flux component in \( r \)-direction \( W/m^2 \)
\( q_x \) heat flux component in \( x \)-direction \( W/m^2 \)
\( Q \) heat source due to other reasons \( W/m^3 \)
\( Q_m \) metabolic heat generation \( W/m^3 \)
\( r \) coordinate variable in radial direction \( m \)
\( r_0 \) radius of the tissue cylinder \( m \)
\( r_p \) radius of the heating spot \( m \)
\( r \) position vector \( m \)
\( t \) time \( s \)
\( t_p \) duration time of the heating process \( s \)
\( T \) tissue temperature \( K \)
\( T_o \) initial tissue temperature \( K \)
\( T_b \) blood temperature \( K \)
\( w_b \) blood perfusion rate \( m^3/(m^2{\cdot}t) \)
\( x \) coordinate variable in axial direction \( m \)
\( x_o \) thickness of the cylindrical tissue \( m \)

\( \alpha \) thermal diffusivity \( m^2/s \)
\( \alpha \tau \) phase lag of the temperature gradient \( s \)
\( \alpha \tau_c \) phase lag of the heat flux \( s \)
\( \beta \) thermal relaxation time in hyperbolic model \( s \)
\( \delta r \) distance between two neighboring grid points in \( r \)-direction \( m \)
\( \delta x \) distance between two neighboring grid points in \( x \)-direction \( m \)
\( \rho \) tissue mass density \( kg/m^3 \)
\( \rho_b \) blood mass density \( kg/m^3 \)
\( \rho_c \) coordinate variable in axial direction \( m \)
\( \tau_b \) thermal relaxation time in hyperbolic model \( s \)
\( \tau_c \) phase lag of the heat flux \( s \)
\( \tau_t \) size of each control volume in \( r \)-direction \( m \)
\( \Delta r \) time step \( s \)
\( \Delta x \) size of each control volume in \( x \)-direction \( m \)

The specific forms of bioheat conduction equation depend upon the constitutive relation between the heat flux \( \mathbf{q} \) and temperature \( T \).

Combining the energy balance equation (4) and the dual-phase-lag constitutive relation (3) while eliminating the heat flux \( \mathbf{q} \) leads to the following DPL bioheat conduction equation for tissue temperature:

\[
\tau_0 \frac{\partial^2 T}{\partial t^2} + \left( 1 + \frac{w_b \rho_b c_b}{\rho c} \tau_q \right) \frac{\partial T}{\partial t} = \alpha \nabla \cdot (\nabla T) + \alpha \tau_t \frac{\partial}{\partial t} \left[ \nabla \cdot (\nabla T) \right] + \frac{w_b \rho_b c_b}{\rho c}(T_b - T)
\]

The DPL bioheat conduction equation can also be derived for heat flux \( \mathbf{q}(\mathbf{r}, t) \) by eliminating the temperature \( T \):

\[
\frac{\partial \mathbf{q}}{\partial t} + \tau_q \frac{\partial^2 \mathbf{q}}{\partial t^2} = \alpha \nabla \cdot (\nabla \mathbf{q}) + \alpha \tau_t \frac{\partial}{\partial t} \left[ \nabla \cdot (\nabla \mathbf{q}) \right] + \alpha \tau_t \phi_b \rho_b \frac{\partial}{\partial t} \left[ (\nabla \cdot (\nabla T)) \right] - \alpha \tau_t \left[ \frac{\partial}{\partial t} \left( \nabla Q_m + \nabla Q \right) \right]
\]

In the derivations of Eqs. (5) and (6), all the thermal properties are assumed to be constant. Each of Eqs. (5) and (6) has more terms compared to their counterpart DPL heat conduction equation for industry materials. Attention is paid to those terms involving the blood perfusion with tissue temperature \( T \). There is no coupling between \( T \) and \( \mathbf{q} \) in the temperature formulation (5), whereas the coupling exists in the heat flux formulation (6).

For numerical analysis, we consider a cylindrical biological tissue of radius \( r_0 \) and thickness \( x_o \) irradiated by electromagnetic waves normally to the upper surface \((x=0)\) over a circular spot of radius \( r_p \) for a time period \( t_p \) (Fig. 1). The living tissue is initially at a uniform temperature \( T_o \). Assume that the irradiation is uniform or in a Gaussian distribution. The thermal load can be modeled as a surface heat flux or a body heat source, depending on the irradiation penetration depth. Let the center of the incident heat flux \( q_p(r) \) coincide with the origin \( O \) of the cylindrical coordinate system. The other boundary surfaces are assumed to be thermally insulated. Due to the axisymmetry of the tissue geometry and the thermal load, only a 2D axisymmetric cylindrical domain \( \Omega ABC \) (Fig. 1) needs to be analyzed for the thermal transfer in the tissue.

In this study, the heat flux formulation of the DPL bioheat conduction, i.e., Eq. (6), is adopted for analyzing the heat conduction in the living tissue shown in Fig. 1 since it is more convenient for problems involving only heat flux boundary conditions. Although blood temperature \( T_b \) and the metabolic heat generation \( Q_m \) may vary with time and space, they are assumed to be constant here for the purpose of the calculation.
emphasizing the essential physics of the DPL bioheat transfer. Under this assumption, those terms with the gradients of $T_b$ and $Q_m$ can be dropped off from Eq. (6). Accordingly, the two axisymmetric equations governing the heat transfers in the $r$- and $x$-direction can be expressed from Eq. (6):

\[
\frac{\partial q_r}{\partial t} + \tau_r \frac{\partial^2 q_r}{\partial t^2} = \alpha \frac{\partial}{\partial r} \left[ \frac{1}{r} \left( \frac{\partial (rq_r)}{\partial r} + \frac{\partial (rq_x)}{\partial x} \right) \right] \\
+ \alpha \tau_r \frac{\partial^2}{\partial r \partial t} \left[ \frac{1}{r} \left( \frac{\partial (rq_r)}{\partial r} + \frac{\partial (rq_x)}{\partial x} \right) \right] \\
+ \alpha w_b \rho_b c_b \frac{\partial T}{\partial r} + \alpha w_b \rho_b c_b \frac{\partial^2 T}{\partial r^2} - \alpha \frac{\partial Q}{\partial r} \\
- \alpha \tau_r \frac{\partial^2 Q}{\partial r \partial t} \\
\frac{\partial q_x}{\partial t} + \tau_q \frac{\partial^2 q_x}{\partial t^2} = \alpha \frac{\partial}{\partial x} \left[ \frac{1}{r} \left( \frac{\partial (rq_r)}{\partial r} + \frac{\partial (rq_x)}{\partial x} \right) \right] \\
+ \alpha \tau_r \frac{\partial^2}{\partial x \partial t} \left[ \frac{1}{r} \left( \frac{\partial (rq_r)}{\partial r} + \frac{\partial (rq_x)}{\partial x} \right) \right] \\
+ \alpha w_b \rho_b c_b \frac{\partial T}{\partial x} + \alpha w_b \rho_b c_b \frac{\partial^2 T}{\partial x^2} - \alpha \frac{\partial Q}{\partial x} \\
- \alpha \tau_r \frac{\partial^2 Q}{\partial x \partial t} \\
\] (7)

Eqs. (7) and (8) will be solved with proper initial and boundary conditions. In this study, the initial conditions of the heat flux and its time-rate are assumed to be zero,

\[
q_r = q_x = 0, \quad \frac{\partial q_r}{\partial t} = \frac{\partial q_x}{\partial t} = 0
\]

for $0 \leq r \leq r_0$ and $0 \leq x \leq x_0$ when $t = 0$ (9)

The boundary conditions are given by:

\[
q_r = q_p(r) \quad \text{for} \quad 0 \leq r \leq r_p \quad \text{and} \quad x = 0 \quad \text{when} \quad 0 < t \leq t_p \\
q_r = 0 \quad \text{for} \quad r_p < r \leq r_0 \quad \text{and} \quad x = 0 \quad \text{when} \quad t > t_p \\
q_x = 0 \quad \text{for} \quad 0 \leq r \leq r_0 \quad \text{and} \quad x = x_0 \quad \text{when} \quad t > 0 \\
q_r = 0 \quad \text{for} \quad 0 \leq x \leq x_0 \quad \text{and} \quad r = 0 \quad \text{when} \quad t > 0 \\
q_t = 0 \quad \text{for} \quad 0 \leq x \leq x_0 \quad \text{and} \quad r = r_0 \quad \text{when} \quad t > 0
\] (10)

The first equation of Eq. (10) is for the case of surface heating. It is changed to

\[
q_r = 0 \quad \text{for} \quad 0 \leq r \leq r_p \quad \text{and} \quad x = 0 \quad \text{when} \quad 0 < t \leq t_p
\]

when a body heating is considered.

The control volume method [18] is employed to solve the governing equations (7) and (8) together with the initial and boundary conditions (9)–(11). The derivation of the discretizing equations and numerical algorithm are given in Appendix A.

3. Validation of computer code

The solution of the 2D axisymmetric DPL numerical model is first validated with a test problem. The simulation model is the same as that shown in Fig. 1. The difference is that the upper surface of the cylinder is subject to a first kind of boundary condition rather than laser heating. To keep consistency with the analytical solution presented in [13], the numerical simulation is performed based on dimensionless quantities. The thickness and the radius of the cylinder are $x_0 = 1$ and $r_0 = 2$, respectively. Initially, the cylinder is uniformly at the temperature of $T_0 = 0$. From time $t \geq 0$, the temperature of a finite circular area (whose center is located at $(0, 0)$) at the upper surface of the cylinder is elevated to $T_b = 1$. The radius of the finite heating area is denoted by $r_p$, which changes from 0.05 to 1.0. Other boundaries of the cylinder remain at zero temperature gradient. The temperature formulation of the 2D DPL model (i.e., Eq. (5)) will be used to simulate this test problem. The blood perfusion and metabolic heat generation are turned off in order to make our numerical simulation results comparable with analytical solution [13]. Other simulation parameters (which are also non-dimensional) are as follows: thermal diffusivity $\alpha = 1$, phase lag time of heat flux $\tau_q = 0.05$, phase lag time of temperature $\tau_T = 0.001$.

Fig. 3 shows the spatial distribution of temperature along the cylinder centerline ($r = 0$) at time $t = 0.05$. It can be clearly seen that the 2D temperature distribution gradually approaches the 1D analytical solution [13] when the heating spot size $r_p$ is increased from 0.05 up to 1.0. When the heating spot size becomes equal to the thickness of the cylinder (i.e., $r_p = 1.0$, curve 2 in Fig. 3), the calculating results are almost the same as the analytical solution [13]. This indicates that when the heating spot radius is increased up to the thickness of the cylinder, the 2D DPL heat conduction in the cylinder is close to 1D heat conduction. The calculating results presented in Fig. 3 justify the credibility and reliability of the 2D DPL numerical model developed in this study.

4. Results and discussions

When the tissue temperature is elevated to as high as 100°C, phase change will take place and the tissue can be vaporized. During this stage, pure heat conduction models may not be able to
describe the complex heat transfer process. The simulation parameters employed in this study are chosen to make sure that the tissue temperature will not exceed 100°C. The cases studied in this work may find its potential applications in laser hyperthermia or laser coagulation therapy.

Unless otherwise mentioned, the following properties [19] of a biological tissue are used in the following numerical analysis: \( \rho = 1000 \text{ kg/m}^3 \), \( k = 0.628 \text{ W/(m K)} \) for the thermophysical properties; \( \rho_b = 1.06 \times 10^3 \text{ kg/m}^3 \), \( c_b = 3860 \text{ J/(kg K)} \), \( w_b = 1.87 \times 10^{-3} \text{ m}^3/\text{m}^3 \text{ tissue/s} \), \( T_b = 37 \degree \text{C} \). \( Q_m = 1.19 \times 10^3 \text{ W/m}^3 \) for the thermal physical properties and temperature of blood and the metabolic heat generation, respectively. The two phase lag times \( \tau_T \) and \( \tau_q \) are ranged from 0.5–32 s. Both the thickness and the radius of the cylindrical tissue are 5 cm, and the initial temperature \( T_o \) is 37°C. The heating lasts for 1 s over a circular spot of radius \( r_p = 1-5 \text{ cm} \).

Convergence study of the grid mesh and time step is first carried out. The two phase lag times \( \tau_T \) and \( \tau_q \) are: \( \tau_q = 16 \text{ s} \), \( \tau_T = 2 \text{ s} \). Both the thickness and the radius of the cylindrical tissue are 5 cm, and the initial temperature \( T_b \) is 37°C. The heating lasts for 1 s over a circular spot of radius \( r_p = 1 \text{ cm} \). Fig. 4 shows the transient variations of tissue temperature at the point (0, 0) for different combinations of time step and mesh density. Fig. 4(a) shows the influence of mesh density while time step keeps at \( \Delta t = 0.01 \text{ s} \), and Fig. 4(b) plots the influence of time step while mesh density keeps at 120 \( \times \) 120. As can be seen in Fig. 4(a), when the mesh density is increased up to the value 120 \( \times \) 120, continuing increase in mesh density cannot result in further improvement of the calculating accuracy. It can be seen in Fig. 4(b) that when the time step is reduced down to the value 0.01 s, more decrease in time step cannot lead to further improvement of the calculating precision. Therefore, a mesh of 120 \( \times \) 120 grid points and a time step of 0.01 s are believed to be adequate to give satisfactory results. Thus, they are used in all the simulations in this paper.

Fig. 5 shows the time history of tissue temperature at the point (0, 0) obtained from the present DPL bioheat conduction model for \( \tau_q = 16.0 \text{ s} \) and different values of \( \tau_T \). In this calculation, the boundary condition of the second kind (the first equation of Eq. (10)) is applied for a surface heating with a constant heat flux \( q_p = 4 \text{ W/cm}^2 \) and \( r_p = 1 \text{ cm} \). In Fig. 5, the curve 5 is the result obtained from the thermal wave model (\( \tau_T = 0 \)), for which the temperature rise is greatest, followed by several noticeable oscillations. The thermal wave effect is diminished as the lagging time \( \tau_T \) increases. For clarity, the curves in the time interval 0–5 s are zoomed out in Fig. 5(b). As previously studied [12], the phase lag \( \tau_T \) is a measure of the time delay in conduction that occurs along microscopic paths (e.g., within meat particles), which cannot be captured by the classical Fourier approach during nonequilibrium. This delay tends to smooth the sharp wave-fronts by promoting conduction, leading to a non-Fourier diffusion-like conduction. As a result of the phase lag \( \tau_T \), the DPL bioheat conduction model...
would damp the thermal wave response. When the phase lag \( \tau_T \) is larger than \( \tau_q \), an over-diffusion phenomenon takes place [20], leading to a much lower temperature rise (curve 1 in Fig. 5).

It is interesting to note that the temperature computed with \( \tau_q = \tau_T = 16.0 \, s \) (curve 2 in Fig. 5) differs from that computed with the Pennes’ bioheat conduction equation (\( \tau_q = \tau_T = 0 \), curve 6 in Fig. 5). This is in contrast to the traditional DPL heat conduction model for industry materials, for which it approaches the Fourier heat conduction model different from that of the traditional DPL heat conduction model. The discrepancy can be attributed to the blood perfusion leads to an extra term involving with \( \tau_T \) in the temperature formulation (the second term in the parenthesis on the left-hand side of Eq. (5)) and two extra terms involving with \( \tau_T \) in the heat flux formulation (the fourth and sixth terms on the right-hand side in Eq. (6)). These make the intrinsic characteristic of the DPL bioheat conduction model different from that of the traditional DPL heat conduction model.

Fig. 6 displays the temperature at the point (0, 0) as a function of time for \( \tau_T = 0.5 \, s \) and different \( \tau_q \)'s. The incident heat flux is \( q_p = 4 \, W/cm^2 \), and the radius of the heated spot is \( r_p = 1 \, cm \). It can be seen from Figs. 7(a)–(d) that the thermal wave response, simulated by the DPL bioheat conduction model, occurs in both the axial (Fig. 7(b)) and radial (Figs. 7(b)–(d)) directions. On the other hand, the classical Pennes’ bioheat model leads to smooth temperature distributions (Figs. 7(e)–(f)). The transients of the tissue temperature, particularly in the region near the heated spot edge, shown in Fig. 7 manifest the inadequacy of the 1D approach.

Fig. 8 shows the transients of temperature at the point (0, 0) for the tissue subjected to a body heating. In this simulation, a uniform irradiation of intensity \( q_p = 150 \, W/cm^2 \) is impinging on the top surface \( x \) over a circular spot of \( r_p = 2 \, cm \). The volumetric heat source is assumed to follow the Beer’s law, \( Q(x, r) = \mu_a \cdot q_p \exp(-\mu_S x) \) with the effective absorption coefficient \( \mu_a = 1 \, cm^{-1} \). It can be seen from Fig. 8 that the impact of the phase lags on the temperature response at point (0, 0) is not so pronounced as that found in the case of surface heating (Figs. 5 and 6). The maximum temperature rises are almost the same for all the combinations of the phase lag times \( \tau_T \) and \( \tau_q \) studied here.

The foregoing simulations are performed for tissues that are heated by a uniform incident heat flux. In real medical practices, the irradiations can be in Gaussian distribution. Fig. 10 illustrates the 2D temperature distributions at \( t = 10 \, s \) for the tissue heated by an incident Gaussian heat flux \( q_p = q_{p\text{max}} \exp(-r^2/r_p^2) \), where \( q_{p\text{max}} = 4 \, W/cm^2 \) and \( r_p = 1 \, cm \) for the surface heating (Fig. 10(a)) and \( q_{p\text{max}} = 150 \, W/cm^2 \) and \( r_p = 2 \, cm \) for the body heating (Fig. 10(b)). For the latter, the Beer’s law is used again to describe the volumetric heat source distribution in the axial di-
Fig. 7. 2D temperature distributions at different time instants. (a)-(d) DPL model; (e)-(f) Pennes model.

As shown in Fig. 10, no thermal wave exhibits in the body heating case but in the surface heating case. Due to the non-uniform distribution of the incident heat flux, the 2D effect on the temperature transient is pronounced for both cases.

5. Conclusions

The DPL bioheat conduction equations in a 2D axisymmetric living tissue are derived for thermal transport in living biological tissues that incorporates the blood perfusion, metabolism and
Fig. 8. Temperature transients at the point (0, 0) for different combinations of $\tau_T$ and $\tau_q$. (a) Influence of $\tau_T$ while $\tau_q$ keeps unchanged; (b) influence of $\tau_q$ while $\tau_T$ keeps unchanged.

Fig. 9. 2D temperature distributions at different time instants for body heating. (a) $t = 1$ s; (b) $t = 10$ s; (c) $t = 20$ s; (d) $t = 40$ s.

other volumetric heat generations. Two local heating conditions, surface heating and body heating, are examined. The effects of the two phase lag times $\tau_T$ and $\tau_q$ on temperature transients are investigated. The numerical results show that the DPL bioheat conduction model describes different thermal responses from the thermal-wave and classical Pennes' bioheat models, depending on the values of the two lagging times. Due to the presence of the blood perfusion in living tissues, the DPL bioheat conduction model can reduce to the Pennes’ bioheat conduction model only when $\tau_q = \tau_T = 0$. This is in contrast to the traditional DPL heat conduction model for which the condition $\tau_q = \tau_T$ (not necessarily equal to zero) is sufficient for it to reduce to the Fourier heat conduction model. The thermal wave effect is diminished as the lagging time $\tau_T$ increases. An over-diffusion could occur when $\tau_T$
is larger than \( r_s \). For local surface heating with a uniform incident heat flux, thermal wave exhibits in both the axial and radial directions, while it could only exhibit in the radial direction for a body heating. When the spot of a local heating is smaller than the tissue bulk size, the 2D effect may become pronounced, particularly for the region near the spot edge.

**Appendix A**

The control volume method [18] is employed to solve the governing equations (7) and (8) together with the initial and boundary conditions (9)–(11). The entire computational zone is divided into \( L_x \times M_r \) control volumes. Each control volume is represented by a grid point placed at its geometric center. For example, the shaded part shown in Fig. 2 represents a control volume \( P \) with faces \( e, w, n \), and \( s \) connecting to its neighbor control volumes \( E, W, N, \) and \( S \), respectively. The lengths of the control volume are \( \Delta x \) in the \( x \)-direction and \( \Delta r \) in the \( r \)-direction. The distances between two neighboring grid points in the \( x \)-direction are \( (\Delta x)_w \) and \( (\Delta x)_n \), and those in the \( r \)-direction are \( (\Delta r)_n \) and \( (\Delta r)_s \). In general, \( (\Delta x)_w \neq (\Delta x)_n \) and \( (\Delta r)_n \neq (\Delta r)_s \).

Integrating Eq. (7) over a control volume with the grid point \( P \), for example, and over the time step from \( t \) to \( t + \Delta t \) and then discretizing the result based on the backward difference in time and the piecewise-linear profile in space yields [18]:

\[
a_{p}q_{r}^{t+\Delta t}=a_{N}q_{r}^{t+\Delta t}+a_{S}q_{r}^{t+\Delta t}+b_{r}
\]

where

\[
a_{N} = \left( \frac{0.5\Delta x}{r_n} + \frac{\Delta x}{(\partial r)_n} \right)(\Delta t + \tau_T)
\]

\[
a_{S} = -\left( \frac{0.5\Delta x}{r_s} + \frac{\Delta x}{(\partial r)_s} \right)(\Delta t + \tau_T)
\]

\[
a_{p} = \frac{\Delta x\Delta r}{\alpha} \left( 1 + \frac{q_{x}^{t+\Delta t}}{\Delta t} - \frac{q_{x}^{t-\Delta t}}{\Delta t} \right)
\]

\[
b_{r} = \frac{\Delta x\Delta r}{\alpha} \left( 1 + \frac{q_{r}^{t+\Delta t}}{\Delta t} - \frac{q_{r}^{t-\Delta t}}{\Delta t} \right)
\]

In Eqs. (A.1) and (A.5), the symbols \( q_{r}^{t+\Delta t} \) represents the heat flux component \( q_{i} (i=x, r) \) at grid point \( \xi = (P, E, W, N, S, NE, NW, SE, SW) \) at the time step \( \zeta = (t - \Delta t, t + \Delta t) \); \( T_{r}^{t+\Delta t} \) represents the temperature at grid point \( \xi = (N, S) \) at the time step \( \zeta = (t, t + \Delta t) \); \( r_n \) and \( r_s \) are the radii of the interfaces \( n \) and \( s \), respectively.

Similarly, the equation governing the heat flux component in the \( x \)-direction can be discretized as follows:

\[
a_{p}q_{x}^{t+\Delta t}=a_{E}q_{x}^{t+\Delta t}+a_{W}q_{x}^{t+\Delta t}+b_{x}
\]

where

\[
a_{E} \equiv \frac{\Delta r}{(\Delta x)_e}(\Delta t + \tau_T)
\]

\[
a_{W} \equiv -\frac{\Delta r}{(\Delta x)_w}(\Delta t + \tau_T)
\]

\[
a_{p} = a_{E} + a_{W} + \frac{\Delta x\Delta r}{\alpha} \left( 1 + \frac{q_{x}^{t+\Delta t}}{\Delta t} \right)
\]

\[
b_{x} = \frac{\Delta x\Delta r}{\alpha} \left( 1 + \frac{q_{x}^{t+\Delta t}}{\Delta t} \right)
\]

The temperatures \( T \) in Eqs. (A.5) and (A.10) can be approximated by discretizing the bioheat energy equation (4). For example, the temperature at the grid point \( P \) is
\[ T_P^{t+\Delta t} = \frac{1}{\rho c \Delta x \Delta r + w_b \rho_b c_b \Delta t \Delta x \Delta r} \times \left\{ -0.5 \Delta t \Delta r \left[ r_N \left( q_{t_P}^{t+\Delta t} + q_{t_N}^{t+\Delta t} \right) \right] - \Delta t \Delta x \Delta r \left[ T_m + (\rho c \Delta x \Delta r)^2 \right] \right\} \]