Endogenous Credit Cycles*

Chao Gu
University of Missouri-Columbia

Fabrizio Mattesini
University of Rome Tor-Vergata

Cyril Monnet
University of Bern and
Study Center Gerzensee

Randall Wright
University of Wisconsin - Madison,
FRB Minneapolis, FRB Chicago and NBER

September 7, 2012

Abstract

We study models of credit with limited commitment and, hence, endogenous borrowing constraints. The baseline model always has multiple stationary equilibria, and nonstationary equilibria where debt limits converge to 0 over time. There can exist equilibria where credit conditions display deterministic cyclic or chaotic dynamics, and stochastic (sunspot) equilibria, where credit conditions fluctuate even though fundamentals are deterministic and time invariant. The results hold when the terms of trade are determined by Walrasian pricing or by generalized Nash bargaining, although the economic forces driving endogenous fluctuations in the two versions are very different. Examples and applications are discussed.

JEL classification Nos: E20, E32
Key words: credit, debt, cycles, dynamics, commitment

*We thank two referees and the editor (Robert Shimer) for very useful comments and suggestions. We also thank Huberto Ennis, Karl Shell and participants at many seminars for their input. Fernando Alvarez, Michael Woodfood and Christian Helwig in particular provided some very useful input during and after seminars at Chicago, Columbia and UCLA. For research support, Wright thanks the NSF and the Ray Zemon Chair in Liquid Assets at the Wisconsin School of Business. The usual disclaimers apply. Corresponding author: Chao Gu, 118 Professional Building, Department of Economics, University of Missouri, Columbia, MO 65211. Email: guc@missouri.edu. Tel: 573-882-8884. Fax: 573-882-2697.
1 Introduction

Are credit markets susceptible to animal spirits, self-fulfilling prophecies and endogenous fluctuations? This is different from asking if there are channels in credit markets that amplify or propagate fluctuations driven by fundamental factors, as emphasized in the literature surveyed by Gertler and Kiyotaki (2010). In this paper, we show that even if fundamentals are deterministic and time invariant, economies with credit-market frictions can display cyclic, chaotic and stochastic dynamics driven purely by beliefs. The key friction is limited commitment, which leads to endogenous borrowing constraints. While most studies of these kinds of models focus on stationary equilibria, we show that economies with commitment frictions not only generate multiple steady states, with different credit conditions and allocations, they also display equilibria where credit conditions and allocations vary over time, even when fundamentals do not. This is true when the terms of trade are determined by Walrasian pricing or by generalized Nash bargaining, although the economic forces generating dynamics in economies with the two pricing mechanisms are very different.

With limited commitment, agents honor their debt obligations to avoid punishment by exclusion from future credit. If one believes one’s debt limit in the future will be 0, one has nothing to lose by reneging on current obligations, which makes the equilibrium debt limit 0 today. Hence there is always a no-credit equilibrium. Generally, there is also a steady state equilibrium with a positive debt limit. To explain how we get cycles, we find it useful to first build some intuition by considering a competitive labor market. Let the unconstrained equilibrium hours and wages be \((\ell^*, w^*)\). Suppose we impose an exogenous restriction \(\ell \leq \phi\), where \(\phi = \ell^* - \varepsilon\) with \(\varepsilon > 0\). There are two effects on workers: \(\ell\) goes down and \(w\) goes up. The first effect has a second-order impact on their utility, by the envelope theorem, if \(\varepsilon\) is

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1By modeling debt limits endogenously using limited commitment, we follow Kehoe and Levine (1993, 2001) and Alvarez and Jermann (2000). See Azariadis and Kass (2007,2008), Lorenzoni (2008), Hellwig and Lorrenzoni (2009) and Sanches and Williamson (2010) for related work. As discussed in more detail below, our setup differs slightly from the usual limited-commitment model, and is based on a framework we use elsewhere to study banking (Gu et al. 2012). While there are no banks here, the setup is used because it is tractable and flexible. In particular, we have production and investment, because we think it may be interesting to see how these vary over the credit cycle, but as we show, the model can be reinterpreted as a Kehoe-Levine pure-endowment economy, and as some other standard credit models.
not too big. The increase in \( w \), however, generates a first-order increase in utility. This is why unions want to increase \( w \) and lower \( \ell \), compared to a competitive market. Of course, no individual worker can affect \( w \) in a competitive market, but an outside (e.g., union or policy) restriction on \( \ell \) can generate a wage-increasing effect.\(^2\)

Now consider a competitive credit market. No individual borrower can affect the terms of his loan by taking on less debt than his unconstrained equilibrium choice, say \( y^* \). But if all borrowers face a constraint \( y \leq \phi = y^* - \varepsilon \), as long as \( \varepsilon \) is not too big, they are all better off, because the improvement in the terms of trade implies a first-order increase in utility, while the impact of the reduction in \( y \) is second order, again by the envelope theorem. Once this is understood, credit cycles emerge as follows. If you believe the debt limit tomorrow will decrease somewhat, but not too much, your utility will increase tomorrow. This makes you more reluctant to renege on current debt, because you do not want to be shut out of tomorrow’s market. The reluctance to renege makes today’s endogenous debt limit high. Tomorrow, you believe the debt limit the next day will go back up. This makes you worse off the day after tomorrow, again based on the envelope theorem. So tomorrow you are more inclined to renege and this makes tomorrow’s debt limit low. There is thus a natural tendency for credit conditions to go up and down – a two-period cycle. Higher-order cycles, chaotic dynamics and sunspot equilibria are more complicated versions of the same idea.

A bilateral bargaining model can generate similar results for completely different reasons. Generalized Nash bargaining has the property that the surplus of both parties need not increase with the total surplus: one agent’s surplus must rise but the other’s may fall (see Aruoba, Rocheteau and Waller 2007 for a recent discussion). When a borrower is negotiating over the terms of a loan, he can therefore be worse off if his debt limit goes up. This is different from the envelope-theoretic reasoning in Walrasian markets, but still implies credit cycles – if your debt limit goes up you can be worse off when negotiating tomorrow’s credit contract, justifying a low limit today, etc. Note that there cannot be cycles in the model with take-it-or-leave-it offers by the borrower, or with Kalai’s proportional bargaining solution, both of

\(^2\)As suggested by a referee, we note that one can provide related examples from international trade theory using tariffs; we do not, to save space, and because we think the labor example suffices to make the point.
which imply the borrower’s surplus must increase with his debt limit. But with generalized
Nash bargaining, as is the case with Walrasian pricing, borrowers can be better off when debt
limits tighten. In either case, in contrast to the labor market example, in credit markets
we do not need outside forces like unions or government to impose the relevant quantity
restriction: limited commitment delivers this as an equilibrium outcome.

The rest of the paper involves making this precise and providing the proofs. Before
beginning, we mention that, as is well known, similar dynamics emerge in monetary models,
and in particular the no-credit equilibrium is reminiscent of the non-monetary equilibrium
that always exists with fiat currency. The economic forces are different here. At the risk of
oversimplifying, the basic mechanism at work in those models is a backward-bending savings
function. The key economic mechanism at work in our model – payoffs decreasing as quantity
constraints are relaxed – is different.  

2 The Model

Time is discrete and continues forever. Each period has two subperiods. There are equal
measures of two types of agents. Type 1 agents consume and type 2 agents produce a good $x$
in the first subperiod. Type 1 agents produce an intermediate good in the first subperiod and
invest it, to generate a return $y$ that type 2 likes to consume in the second subperiod. There
are gains from trade using the following credit relationship: type 1 agents (borrowers) get $x$
now from type 2 agents (lenders), and promise to deliver $y$ later, when their investments come
to fruition. Only type 1 have access to the investment opportunity. The utility from this
arrangement is $U^1(x, y)$ for type 1 and $U^2(y, x)$ for type 2, where $U^j$ is strictly increasing in
the first argument, which is $j$’s consumption, and strictly decreasing in the second, which is
$j$’s production. Utility is also strictly concave, twice differentiable, and satisfies $U^j(0, 0) = 0$.

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3 See Azariadis (1993) for a textbook treatment, and references to primary sources, on money and dynamics in the overlapping-generations tradition; see Woodford (1986,1988) for models with cash-in-advance or borrowing constraints; and see Matsuyama (1990) for money in the utility function. Boldrin and Woodford (1990) provide a survey of related work. Matsuyama (2012) provides an alternative approach. There is also work on dynamics in growth models with externalities (e.g., the Benhabib 1992 volume); there are no such externalities in our model.

4 Sanches and Williamson (2010) also comment on a similarity between money and credit models, and particularly on the no-credit equilibrium, but they do not study dynamics.
We also assume normal goods for some results.

The key friction in the model is limited commitment: rather than honor his obligation to deliver $y$ in the second subperiod, type 1 may abscond with or otherwise divert the investment returns for his own purposes, as in Biais et al. (2007) or DeMarzo and Fishman (2007). If type 1 behaves opportunistically in this way, he gets a payoff $\lambda y$ over and above $U^1(x, y)$, where $\lambda$ parameterizes the temptation to renege. We impose $U^1(x, y + y') + \lambda y' \leq U^1(x, y)$ for all $x, y, y' \geq 0$. This says that it is never efficient ex ante for type 1 to divert $y'$ (he is better off not producing/investing it in the first place). However, he may be tempted to divert resources ex post, when the production cost is sunk. So trade must be self enforcing: $U^j$ must be nonnegative for both agents, and we have to ensure compliance by type 1 in the second subperiod. The incentive to honor his obligation comes from the threat to exclude type 1 from future credit, which gives him an autarky payoff normalized to 0. However, because it is useful in applications and examples, we allow imperfect monitoring: a defaulting type 1 only gets caught with probability $\pi$.\footnote{As mentioned, this environment is taken from Gu et al. (2012), where we discuss imperfect monitoring in more detail, but there we only study the incentive-feasible set, in the spirit of mechanism design, while here we study price-taking and bargaining equilibria. To be clear, we do not need $\pi < 1$ for any general results, but it can be useful in applications and examples.}

Let $V^j_t$ be type $j$’s value function at $t$ given an allocation $(x_t, y_t)$, which we call a credit contract, since it specifies that type 1 gets $x$ now and repays $y$ later. Let $\beta \in (0, 1)$ be the discount factor across periods (discounting across subperiods is subsumed in the $U^j$ notation). Then

\begin{align}
V^1_t &= U^1(x_t, y_t) + \beta V^1_{t+1} \\
V^2_t &= U^2(y_t, x_t) + \beta V^2_{t+1}.
\end{align}

A feasible contract must satisfy the participation constraints in the first subperiod,

$$U^1(x_t, y_t) \geq 0 \text{ and } U^2(y_t, x_t) \geq 0,$$

but these never bind. The critical condition is the repayment constraint for type 1 in the second subperiod,

$$\lambda y_t + (1 - \pi) \beta V^1_{t+1} \leq \beta V^1_{t+1}.$$
The LHS of (4) is 1’s instantaneous deviation payoff $\lambda y_t$, plus the continuation value: with probability $\pi$ he is caught and excluded from future markets; with probability $1 - \pi$ he is not and continues in good standing. We rewrite (4) as

$$y_t \leq \phi_t,$$

where $\phi_t \equiv (\beta \pi / \lambda) V^1_{t+1}$ is the debt limit. Using (1), we can express this recursively to make it clear the debt limit in one period depends on the debt limit in the next period,

$$\phi_{t-1} = \frac{\beta \pi}{\lambda} U^1(x_t, y_t) + \beta \phi_t.$$

We consider both competitive markets and search-and-matching markets, but to ease the presentation, we present only the former in the text and relegate the latter to Appendix A. Thus, each period, everyone is assigned to one of a large number of spatially distinct Walrasian markets, where they trade short-term (across subperiod) credit contracts taking prices as given.\(^6\) Letting $y$ be numeraire, type 1 maximizes utility given his budget and credit constraints:

$$\max_{x_t, y_t} U^1(x_t, y_t) \text{ s.t. } p_t x_t = y_t \text{ and } (5)$$

Type 2, with no repayment issues, solves

$$\max_{y_t, x_t} U^2(y_t, x_t) \text{ s.t. } p_t x_t = y_t$$

Let $(x^*, y^*)$ denote equilibrium ignoring the repayment constraint, the solution to

$$U^1_x (x_t, y_t) x_t + U^1_y (x_t, y_t) y_t = 0 \quad \text{ and } \quad U^2_x (y_t, x_t) x_t + U^2_y (y_t, x_t) y_t = 0.$$  

If $y^* \leq \phi_t$ we can implement the unconstrained allocation $(x^*, y^*)$. Otherwise equilibrium is constrained: $y_t = \phi_t$ and $x_t = h(\phi_t)$ is the solution to (10) with $y_t = \phi_t$.

\(^6\)There are no long-term trades. One way to motivate this is to say that agents meeting in one market never meet again, as formalized in Aliprantis, Camera and Puzzello (2006,2007) – this is why we assume a large number of segmented markets. It is even easier to motivate the lack of long-term credit contracts in the search-and-bargaining version of the model – where agents trade bilaterally, and never meet again, with probability 1. Also, to avoid issues concerning renegotiation, or the incentive compatibility of punishments, when we say deviators are excluded from future credit markets we mean they literally lose access to these markets, and are simply not allowed in. We also emphasize that do not allow entry: the measures of borrowers and lenders (types 1 and 2) are fixed. Some results could change if one allows entry.
Since \( x^* = h(y^*) \), equilibrium can be represented as:

\[
\begin{align*}
\text{if } \phi_t < y^* & \text{ then } x_t = h(\phi_t) \text{ and } y_t = \phi_t \\
\text{if } \phi_t \geq y^* & \text{ then } x_t = h(y^*) \text{ and } y_t = y^*
\end{align*}
\]

(11)

It will be useful below to note the following: when \( y_t \leq \phi_t \) is binding, \( x_t \) is not necessarily increasing in \( \phi_t \), since

\[
\frac{\partial x}{\partial \phi} = h'(\phi) = -\frac{U^2_y + y \left( U^2_{yy} - \frac{U^2_y}{U^2_y U^2_{xy}} \right)}{U^2_x + x \left( U^2_{xx} - \frac{U^2_x}{U^2_x U^2_{xy}} \right)},
\]

(12)
is ambiguous. More importantly,

\[
\frac{\partial U^1}{\partial \phi} = U^1_y \frac{U^2_x - U^1_x U^2_y - y U^1_x \left( U^2_{yy} - \frac{U^2_y}{U^2_x U^2_{xy}} \right) + x U^1_y \left( U^2_{xx} - \frac{U^2_x}{U^2_x U^2_{xy}} \right)}{U^2_x + x \left( U^2_{xx} - \frac{U^2_x}{U^2_x U^2_{xy}} \right)}
\]

(13)
is ambiguous; as we show below, \( \partial U^1 / \partial \phi < 0 \) is not only possible but inevitable for some \( \phi_t \). Hence, a borrower’s payoff can decrease with his credit limit.

Following Alvarez and Jermann (2000), for all \( t \), the equilibrium debt limit \( \phi_t \) is defined as follows: type 1 is indifferent between repaying \( \phi_t \) and defaulting. In any feasible allocation payoffs, and hence \( \phi_t \), must be bounded (so, as in many other models, we rule out explosive bubbles). We can also bound \( x_t \) and \( y_t \) without loss in generality. Hence we have:

**Definition 1** An equilibrium is given by nonnegative and bounded sequences of debt limits \( \{\phi_t\}_{t=1}^{\infty} \) and credit contracts \( \{x_t, y_t\}_{t=1}^{\infty} \) such that, for all \( t \): (i) \((x_t, y_t)\) solves (11) given \( \phi_t \); (ii) \( \phi_t \) solves (6).

We can collapse the two conditions in Definition 1 into one:

\[
\phi_{t-1} = f(\phi_t) \equiv \begin{cases} 
(\beta \pi / \lambda) U^1 \left[ h(\phi_t), \phi_t \right] + \beta \phi_t & \text{if } \phi_t < y^* \\
(\beta \pi / \lambda) U^1 (x^*, y^*) + \beta \phi_t & \text{otherwise}
\end{cases}
\]

(14)

The dynamical system (14) describes the evolution of the debt limit in terms of itself. This system is forward looking, naturally, in the sense that the debt limit in one period depends on the debt limit in the next period. Equilibria are characterized by nonnegative bounded solutions \( \{\phi_t\} \) to (14), from which one can solve for the allocation using (11). This completes our description of the baseline model.
3 Results

A stationary equilibrium, or steady state, is a fixed point \( f(\phi) = \phi \). Obviously \( \phi = 0 \) is one such point, associated with \((x, y) = (0, 0)\). Intuitively, if there is to be no credit in the future, you have nothing to lose by reneging, so no one will extend you credit, today.

A nondegenerate steady state is a solution to \( f(\phi^*) = \phi^* > 0 \). The mild assumption \( U_1^x(0, 0) h'(0) + U_1^y(0, 0) > \lambda(1 - \beta)/\beta \pi \) guarantees:

**Proposition 1** There exists at least one positive steady state, \( \phi^* = f(\phi^*) > 0 \). If \( f(y^*) < y^* \), the repayment constraint is binding at any steady state.

All proofs are in the Appendix, but the existence result is easy to understand from Figures 1a and 1b (all figures are at the end of the paper). In Figure 1a the debt constraint in steady state is slack, \( \phi^* \geq y^* \), and in Figure 1b it is binding, \( \phi^* < y^* \). Note that \( f(\phi) \) is not necessarily monotone for \( \phi \in (0, y^*) \), so we cannot guarantee uniqueness of a positive steady state \( \phi^* \), in general, although it was unique in all the examples we tried. Some statements of results below proceed as if \( \phi^* \) were unique, but this is only to ease the presentation, and all results all hold more generally if one replaces the clause “the unique positive” with “the smallest positive” steady state.\(^7\)

It is clear that there always exist equilibria where \( \phi_t \to 0 \), as shown in Figures 2a and 2b. Note that with \( \phi_{t-1} \) on the horizontal and \( \phi_t \) the vertical axis, in these figures the curve is \( f^{-1} \). We draw it this way because the dynamics are forward looking, with causality running from future to previous periods, and we want the dependent variable on the vertical axis. In Figure 2a \( f \) is monotone, so \( f^{-1} \) is a function; in Figure 2b it is not monotone, so \( f^{-1} \) is a correspondence. When \( f \) is monotone, once we pick an initial \( \phi_0 \in (0, \phi^*) \) the sequence \( \{\phi_t\} \) is pinned down. When \( f \) is not monotone, over some range for \( \phi_0 \) there are multiple equilibrium paths for \( \{\phi_t\} \). There are even perfect foresight equilibria where we start at \( \phi^* \), and stay there for any number of periods, until dropping to the lower branch of \( f^{-1} \), and then \( \phi_t \to 0 \). Summarizing this discussion, we have established the following:

\(^7\)One can make a few observations about the case with multiple positive steady states, e.g., they alternate between stable and unstable. We omit this in the interest of space.
Proposition 2 Suppose there is a unique positive steady state $\phi^s$. Let $\tilde{\phi} = \arg \max f(\phi_t)$ s.t. $\phi_t \in [0, \phi^s]$. Starting from any $\phi_0 < \tilde{\phi}$, there is a nonstationary equilibrium, and possibly more than one, with $\phi_t \to 0$.

Now consider two-period cycles, with $\phi^1$ and $\phi^2 > \phi^1$ denoting the periodic points. Following standard methods (e.g., Azariadis 1993), we have:

Proposition 3 Suppose there is a unique positive steady state $\phi^s$. If $f'(\phi^s) < -1$, there is a two-period cycle with $\phi^1 < \phi^s < \phi^2$.

We illustrate the previous result with examples using the following functional forms:

$$ U^1(x, y) = \frac{x^{1-\alpha}}{1-\alpha} - y \ 	ext{and} \ U^2(y, x) = y - \frac{x^{1+\gamma}}{1+\gamma}. \quad (15) $$

For simplicity here we set $\alpha = 0$. Thus, $U^1$ is linear, but since $U^2$ is nonlinear, interesting dynamics still obtain via $x = h(\phi)$. The parameters are not calibrated to be empirically reasonable, only to illustrate some mathematical possibilities (this is not to say the model couldn’t be calibrated more realistically in future work). Examples 1 and 2 show cycles where $\phi^1 < \phi^2 < y^*$ and where $\phi^1 < \phi^* < y^2$, respectively.

Example 1 Let $\gamma = 2.1$, $\beta = 0.4$, $\pi/\lambda = 6$. Then $\phi^s = 0.7194$ and there is a two-cycle with $\phi^1 = 0.4773 < y^*$ and $\phi^2 = 0.9360 < y^*$. See Figure 3a.

Example 2 Let $\gamma = 0.5$, $\beta = 0.9$, $\pi/\lambda = 10$. Now $\phi^s = 0.9674$ and there is a two-cycle with $\phi^1 = 0.9328 < y^*$ and $\phi^2 = 1.0365 > y^*$. See Figure 3b.

The next example has a three-period cycle (Figure 4a). The existence of a three-cycle implies the existence of cycles of all orders (the Sarkovskii theorem), and chaotic dynamics (the Li-Yorke theorem). Chaos is observationally equivalent to a stochastic process for $\phi_t$ (Figure 4b), so debt limits and allocations can appear random when they are not.

Example 3 Let $\gamma = 0.8$, $\beta = 0.9$, $\pi/\lambda = 15$. Now $\phi^s = 0.9835$, and there is a three-cycle with $\phi^1 = 0.9464 < y^*$, $\phi^2 = 1.0516 > y^*$ and $\phi^3 = 1.1684 > y^*$.

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8We return to $\alpha > 0$ below. Note that $\alpha = 0$ provides a stark illustration of the economic mechanism at work: in this case, type 1 gains from trade only if he is constrained.
The next result says that in any cycle we must have \( \phi_t < y^* \) at some point over the cycle, but not necessarily at all points, i.e., the debt limit binds infinitely often.

**Proposition 4** Suppose there is a unique positive steady state \( \phi^s \). In any \( n \)-period cycle, at least one periodic point has the repayment constraint binding, \( \phi_t < y^* \).

The model can also generate stochastic (sunspot) cycles, where \( \phi_t \) and \((x_t, y_t)\) fluctuate randomly. Consider a Markov sunspot variable \( s_t \in \{1, 2\} \), which does not affect fundamentals, but may affect equilibrium. Let \( \Pr (s_{t+1} = 1|s_t = 1) = \sigma_1 \) and \( \Pr (s_{t+1} = 2|s_t = 2) = \sigma_2 \). The economy is in state \( s \) at date \( t \) if \( s_t = s \). Let \( V^j_{s,t} \) be type \( j \)'s value function in state \( s \) at date \( t \), and let \((x_{s,t}, y_{s,t})\) be the state-contingent credit allocation. Then

\[
V^1_{s,t} = U^1 (x_{s,t}, y_{s,t}) + \beta \left[ \sigma_s V^1_{s,t+1} + (1 - \sigma_s) V^1_{-s,t+1} \right] \tag{16}
\]

\[
V^2_{s,t} = U^2 (y_{s,t}, x_{s,t}) + \beta \left[ \sigma_s V^2_{s,t+1} + (1 - \sigma_s) V^2_{-s,t+1} \right]. \tag{17}
\]

The generalized repayment constraint is

\[
\lambda y_{s,t} \leq \phi_{s,t} \equiv \beta \pi \left[ \sigma_s V^1_{s,t+1} + (1 - \sigma_s) V^1_{-s,t+1} \right]. \tag{18}
\]

Equilibrium in state \( s \) at date \( t \) is given by

- if \( \phi_{s,t} < y^* \) then \( x_{s,t} = h (\phi_{s,t}) \) and \( y_{s,t} = \phi_{s,t} \)
- if \( \phi_{s,t} \geq y^* \) then \( x_{s,t} = h (y^*) \) and \( y_{s,t} = y^* \) \tag{19}

and the dynamical system is

\[
\phi_{s,t-1} = \sigma_s \left[ \frac{\beta \pi}{\lambda} U^1 (x_{s,t}, y_{s,t}) + \beta \phi_{s,t} \right] + (1 - \sigma_s) \left[ \frac{\beta \pi}{\lambda} U^1 (x_{-s,t}, y_{-s,t}) + \beta \phi_{-s,t} \right]
\]

\[
= \sigma_s f (\phi_{s,t}) + (1 - \sigma_s) f (\phi_{-s,t}). \tag{20}
\]

**Definition 2** A sunspot equilibrium is given by nonnegative and bounded sequences of debt limits \( \{\phi_{s,t}\}_{t=1, s=1}^\infty \) and contingent contracts \( \{x_{s,t}, y_{s,t}\}_{t=1, s=1}^\infty \) such that, for all \( t \) and \( s \): (i) \((x_{s,t}, y_{s,t})\) solves (19) given \( \phi^*_t \); (ii) \( \phi_{s,t} \) solves (20).

A proper sunspot equilibrium has \( \phi_{1,t} \neq \phi_{2,t} \) for some \( t \). Consider equilibria that depend only on the state, not the date, with \( \phi_2 > \phi_1 \). Then the repayment constraint is binding
in state 1 (otherwise, we have $x^s = x^*$ and $y^s = y^*$ for both states, which implies $\phi_1 = \phi_2$). Following standard methods (again see Azariadis 1981) one can show proper sunspot equilibria exist when $f'(\phi^s) < -1$, the same condition for two-period cycles, and so the examples with cycles also have sunspot equilibria. Formally, we have:

**Proposition 5** Suppose there is a unique positive steady state $\phi^s$. If $f'(\phi^s) < -1$ then there exist $(\sigma_1, \sigma_2)$, $\sigma_1 + \sigma_2 < 1$, such that the economy has a proper sunspot equilibrium in the neighborhood of $\phi^s$.

4 Discussion

The existence of equilibria with deterministic or stochastic cycles relies on the nonmonotonicity of $f(\phi_t)$. To understand this, recall

$$\phi_t = f(\phi_{t+1}) = \frac{\beta \pi}{\lambda} U^1(x_{t+1}; y_{t+1}) + \beta \phi_{t+1}. \tag{21}$$

An increase in the debt limit at $t+1$ influences the economy at $t$ in two ways. First it directly raises $\phi_t$ through the linear term $\beta \phi_{t+1}$. This effect in isolation works against cycles – e.g., for $\phi_{t+1} > y^*$ the allocation $(x_{t+1}; y_{t+1})$ does not depend on $\phi_{t+1}$, so only the linear term in (21) is operative, and the system $\phi_{t+1} = (1/\beta)\phi_t$ is explosive. But there is a second, nonlinear, effect coming from $U^1(x_{t+1}; y_{t+1})$, which is ambiguous in general, but negative when $\phi_{t+1}$ is near $y^*$.\(^9\)

**Proposition 6** With Walrasian pricing, $\partial U^1[h(\phi), \phi] / \partial \phi < 0$ at $\phi = y^* - \varepsilon$ for some $\varepsilon > 0$ if $y$ is a normal good for type 2. With generalized Nash bargaining, if the bargaining power of the borrower is $\theta < 1$, the same result holds if $y$ is a normal good for both types.

Heuristically, in Walrasian equilibrium a buyer of good $y$ is always better off under the restriction $y \leq y^* - \varepsilon$ for some $\varepsilon > 0$, for the same reason that monopolists produce less than competitive suppliers. Pursuing the monopolist analogy, we do not suggest that $\varepsilon$ has to be

\(^9\)Looking at (21), it appears that cycles are more likely when $\beta$ is small, since this reduces the impact of the linear term, as long as we keep the coefficient on the nonlinear term big by increasing $\pi/\lambda$. It is not quite that simple, however, since the allocation $(x_{t+1}; y_{t+1})$ depends on these parameters.
small — that is merely a sufficient condition for the envelope theorem — and the borrowers may be better off for relatively big \( \varepsilon \), just not too big. While debt limits can make borrowers better off, they cannot make everyone better off. For small \( \varepsilon \) lenders lose, and for sufficiently tight debt limits everyone loses (consider \( \phi = 0 \)).

When the nonlinear term in (21) is negative and big, \( f (\phi_{t+1}) \) can be decreasing. We now present conditions guaranteeing the system satisfies the condition for cycles, \( f' (\phi^*) < -1 \).

We start by defining the elasticity

\[
\eta (\phi) = \frac{\phi}{U^1 [h (\phi), \phi]} \frac{\partial U^1 [h (\phi), \phi]}{\partial \phi}.
\]

**Corollary 1** If \( \eta (y^*) < -1 \) then we can always pick \( \beta \) and \( \pi / \lambda \) to generate cycles.

It is not hard to satisfy this elasticity condition — e.g., if \( U^1 = \log (1 + x) - A \log (1 - y) \) and \( U^2 = \log (1 + y) - A \log (1 - x) \), for \( A \in (0, 1) \), \( \eta (y^*) < -1 \) iff \( A > \hat{A} \), where \( \hat{A} \approx 0.132 \). Basically, \( \eta (y^*) < -1 \) allows us to choose \( \beta \pi / (1 - \beta) \lambda \) close to \( y^*/U^1 (x^*, y^*) \), so that \( \phi^* \) is near \( y^* \), and then pick \( \beta \) and \( \pi / \lambda \) to guarantee \( f' (\phi^*) < -1 \). Although this is not a quantitative paper, we mention that having \( \pi / \lambda \) and \( \beta \) makes it easier to construct interesting examples. This is no surprise, but is still relevant — e.g., in future quantitative work, it may be useful to allow the monitoring probability \( \pi \), which is absent in most related models, to play a role.

To say more about conditions to guarantee cycles, reconsider the functional forms in (15), where now \( \alpha \in [0, 1) \). Also assume \( (1 - \alpha) \gamma > (1 + \gamma) \alpha \). Then we have:

**Corollary 2** Given the parameter restrictions listed above, there is a unique positive steady state \( \phi^* > 0 \), and there are cycles around \( \phi^* \) if

\[
\beta < \frac{\gamma (1 - \alpha) - \alpha (1 + \gamma)}{\alpha + \gamma} \quad \text{and} \quad 1 + \frac{2 + \gamma - \alpha}{\beta (\alpha + \gamma)} < \frac{\pi}{\lambda} < \frac{1 - \alpha}{\alpha - \beta}.
\]

**Corollary 3** As a special case, \( \alpha = 0 \) implies cycles exist if \( \pi / \lambda > 1 + (2 + \gamma) / \beta \gamma \).

While it is hard to identify parameter conditions for a local bifurcation around \( \phi^* \), in general, given these functional forms we get \( f' (\phi^*) < -1 \) when \( \pi / \lambda \) is in the interval defined in Corollary 2. When \( \alpha = 0 \), so that \( U^1 \) is linear, all we need is \( \pi / \lambda > 1 + (2 + \gamma) / \beta \gamma \). In this
case, credit cycles are more likely to emerge when $\beta$ and $\pi/\lambda$ is big. It may be surprising that cycles are more likely when agents are patient, the temptation to default is low, and the monitoring probability is high, but that is what we find in this specification.

The last item to discuss is the economic interpretation. In the baseline model, type 2 produces 1’s consumption good, while type 1 produces intermediate goods, invests, and repays 2 from the proceeds. For concreteness, consider an example suggested by the editor. Agent 1 is a contractor who agent 2 wants to do some work on his house. Agent 1 needs to be paid $x$ in advance; then he invests in supplies, or working capital, generally. There is a cost to 2 to providing $x$, which can be a production cost or an opportunity cost of consuming less himself. Agent 1 promises to deliver home improvements/repairs using his labor $y$, but as usual he is tempted to renege for the opportunistic payoff $\lambda y$. So we impose the constraint $\lambda y + (1 - \pi) \beta V_{t+1}^1 \leq \beta V_{t+1}^1$, saying the contractor prefers to do the work rather renege and risk ruining his reputation, which occurs with probability $\pi$, say because homeowners only report bad behavior probabilistically. Hence, there is a maximum $y = \phi$ that 1 can credibly commit to deliver, and there are equilibria where $\phi_t$ displays complicated dynamics. While we can imagine a variety of other applications, contracting with contractors is one that many people have experienced first hand.10

Although we like having production and investment in the model, similar results apply to endowment economies. Consider a simple Kehoe-Levine economy, where type 1 agents are endowed with one unit of a generic good in the second subperiod, and type 2 with a unit in the first. There is no production or investment. Agents all want to consume in both subperiods. If 1 gets $x_t$ from 2 in the first subperiod, and gives him $y_t$ in the second, payoffs are $U^1 = u(x_t) + v(1 - y_t)$ and $U^2 = u(1 - x_t) + v(y_t)$ – a special case of our general setup. The value functions still satisfy (1) and (2), while repayment constraint becomes $v(1) - v(1 - y_t) \leq \beta \pi V_{t+1} \equiv \phi_t$, or $y_t \leq g(\phi_t)$ with $g(\phi_t) = 1 - v^{-1}[v(1) - \phi_t]$.

10The editor’s suggestion was actually more along the lines of MacLeod and Malcolmson (1993): A firm needs to hire workers before profits are realized, with only the promise of future wages. The firm may renege, but faces punishment, like any other defaulter. Hence, there is an maximum credible wage promise. The main difference in these applications concerns who has a first-mover disadvantage – the household paying a contractor $x$ up front and hoping to receive $y$, or an individual working in advance and hoping to get paid. Again, it is easy to imagine other applications along these lines.
The interpretation of $\phi_t$ here is slightly different, compared to the baseline model, where $\phi_t$ was the maximum credible promise, because the deviation payoff here is nonlinear. Still, following the steps taken above, we get:

$$
\phi_{t-1} = f(\phi_t) \equiv \begin{cases} 
\beta \pi u[h(g(\phi_t))] + \beta \pi v[1 - g(\phi_t)] + \beta \phi_t & \text{if } g(\phi_t) < y^*
\beta \pi u(x^*) + \beta \pi v(1 - y^*) + \beta \phi_t & \text{otherwise}
\end{cases}
$$

(23)

System (23) is similar to (14), except now the debt constraint is $y \leq g(\phi)$, while in the baseline model it is $y \leq \phi$. To get cycles we need $f'(\phi^*) < -1$, which in this version means $g'(\phi)$ is big. Since $g'(y^*) = 1/v'(1 - y^*)$, we need $v'(1 - y^*)$ small. Given this, all the results hold, although the algebra was easier when deviation payoffs and hence the repayment constraints were linear. To be clear, usually pure-credit models have two types, one endowed with 1 in even periods and 0 in odd periods, and vice versa for the other, while we have them endowed this way across subperiods. That is not crucial: Kehoe-Levine uses $U = u(c_t) + \beta u(c_{t+1}) + ...$, while we use $U(x_t, y_t) + \beta U(x_{t+1}, y_{t+1}) + ...$ in general, and $U(x_t, y_t) = u(x_t) + v(y_t)$ as a special case, but one can simply reinterpret two periods in their model as one period in ours, with a different $\beta$.

We can also map our setup into a model of secured lending, as in Kiyotaki and Moore (1997). In the first subperiod type 1 is endowed with $C$ units of a good he likes but 2 does not, while 2 is endowed with $K$ units of a good that can be consumed by both types or used as capital to produce more goods in the second subperiod via technology $F(K)$. For simplicity, $K$ and goods fully depreciate across periods. Type 1 considers all goods perfect substitute, but consumes only in the second subperiod, while 2 values goods in both subperiods. If 1 borrows $x$ from 2 and promises to repay $y$, $U^1 = F(x) - y + c - C$, where $c - C$ is his consumption over and above his endowment (think of him as a profit-maximizing producer/investor). Type 2’s payoff is $U^2 = U^2(K - x, y)$ (think of him as a consumer/saver). To secure a loan, 1 pledges $C$ as collateral: if he delivers $y$ to the lender he gets $C$ back; otherwise he gets $F(x) - (1 - \lambda) y$, as the lender gets only a fraction of what he was promised while he takes the rest, but forfeits his collateral and is excluded from future credit with probability $\pi$.  

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The relevant constraint becomes
\[ F(x_t) - y_t + \beta V_{t+1}^1 \geq F(x_t) - (1 - \lambda) y_t - C + (1 - \pi) \beta V_{t+1}^1, \]
or the following linear transformation of our baseline constraint
\[ y_t \leq C/\lambda + (\beta \pi / \lambda) V_{t+1}^1 \equiv \phi_t. \] (24)

If \( \pi = 0 \), this is a simple model of collateralized lending à la Kiyotaki-Moore; if \( C = 0 \) this a special case of our baseline model. Following the same steps, we now get:
\[
\phi_{t-1} = f(\phi_t) \equiv \begin{cases} 
(\beta \pi / \lambda) F[h(\phi_t)] - (\beta \pi / \lambda) \phi_t + \beta \phi_t + (1 - \beta) C / \lambda & \text{if } \phi_t < y^* \\
(\beta \pi / \lambda) F(x^*) - (\beta \pi / \lambda) y^* + \beta \phi_t + (1 - \beta) C / \lambda & \text{otherwise}
\end{cases} \] (25)

By virtue of (24), the equilibrium credit limit is at least \( C/\lambda \). Hence, we lose the no-credit equilibrium (as in the original Kehoe-Levine model, where they endowed agents with assets that can be confiscated in the event of default). Intuitively, even if there were no future credit, you would repay a debt of \( y < C/\lambda \) today to avoid forfeiting valuable collateral.

We also lose dynamic equilibria converging to a no-credit equilibrium. But that does not affect our main results. As long as \( \pi > 0 \), all the results on cycles continue to hold as long as \( C \) is not too big, simply by continuity (payoffs and behavior over a cycle around the positive steady state do not depend critically on what happens near the origin). We mention this not only to show the flexibility of the framework, but to make the following point:

What generates interesting dynamics is the forward-looking nature of credit markets, since \( \pi = 0 \) implies there is only secured lending, as in Kiyotaki-Moore, and hence no endogenous dynamics.\(^{11}\)

5 Conclusion

We developed a framework to study credit market dynamics. There always exist multiple steady states and multiple dynamic equilibria in the baseline model. There can exist de-

\(^{11}\)There are some complications that we do not go into in the interest of space. Thus, e.g., it matters generally whether someone who reneges on his obligations can be punished by exclusion from all credit in the future, or only from unsecured credit. See He, Wright and Zhu (2012) for an application in the context of housing markets, where reneging can result in the debtor losing his home equity as well as his credit cards. That paper does not study dynamics, however, the way we do in this paper.
terministic, chaotic, and stochastic cycles, where credit conditions fluctuate even though fundamentals are deterministic and time invariant. Even if we add features, such as collateral, that rule out the no-credit equilibrium and dynamics paths that converge to the no-credit equilibrium, credit cycles still exist. The key friction in our economy is limited commitment, although there are other ingredients, including imperfect monitoring. Still, the model is very easy to use. One reason is that equilibrium allocations reduce to a sequence of two-period (or two-subperiod) credit arrangements, even though the economy goes on forever.

The setup is also quite flexible, and can deal with both Walrasian pricing and generalized Nash bargaining. We provided several interpretations, including pure-credit economies as well as those with production and investment. Of course, we made some strong assumptions, including our assumption of perfect foresight, or rational expectations. The goal here was to show that even with this discipline one can get interesting dynamics. Presumably, interesting dynamics can be even more likely with learning, imperfect information and other such complications, although we have not studied this. Also, we assumed no fundamental shocks, to make a stark point. It would be interesting to see if combinations of fundamental shocks and the endogenous dynamics emphasized here might generate empirically plausible credit cycles. This and other quantitative work is left to future research.
Appendix A: Search and Bargaining

Suppose for ease of presentation that each type 1 meets a random type 2 at each date $t$. They negotiate a contract $(x_t, y_t)$ taking as given what happens in all other meetings. Generalized Nash bargaining determines $(x_t, y_t)$. Note that the strategic foundations of Nash bargaining are not straightforward in nonstationary situations; see Coles and Wright (1998), Coles and Muthoo (2003) and Ennis (2001, 2004). Thus, we are taking the Nash solution as a primitive – there is no claim here that it is derived from a strategic bargaining game in the usual manner.

Having said that, there is no presumption that one could not generate interesting dynamics with some strategic bargaining model, such as the ones used in the above-mentioned papers.

Assume type 1 has bargaining power $\theta$, and threat points are given by continuation values. Since the continuation values and threat points cancel, the bargaining outcome solves

$$\max_{(x_t, y_t)} U^1(x_t, y_t)^\theta U^2(y_t, x_t)^{1-\theta} \quad \text{s.t. (5).} \quad (26)$$

If we ignore the repayment constraint, the solution $(x^*, y^*)$ to this problem satisfies

$$\theta U^1_x(x_t, y_t) U^2(y_t, x_t) + (1 - \theta) U^1(x_t, y_t) U^2_x(y_t, x_t) = 0 \quad (27)$$

$$\theta U^1_y(x_t, y_t) U^2(y_t, x_t) + (1 - \theta) U^1(x_t, y_t) U^2_y(y_t, x_t) = 0. \quad (28)$$

Given $\phi_t \geq y^*$, we can implement the unconstrained contract; if $\phi_t < y^*$, let $x_t = h(\phi_t)$ solve (27) with $y_t = \phi_t$. Equilibrium satisfies (11), exactly as in the Walrasian model.

When $y \leq \phi$ binds, we have

$$\frac{\partial x}{\partial \phi} = \frac{-\theta \left( U^1_{xy} U^2 + U^1 U^2_y \right) - (1 - \theta) \left( U^1_{y} U^2_x + U^1 U^2_{xy} \right)}{\theta \left( U^1_{xx} U^2 + U^1 U^2_x \right) + (1 - \theta) \left( U^1_{x} U^2_x + U^1 U^2_{xx} \right)}, \quad (29)$$

which is ambiguous, in general. So is

$$\frac{\partial U^1 [h(\phi), \phi]}{\partial \phi} = \frac{\theta U^2 \left( U^1_{xx} U^1_y - U^1_{xy} U^1_x \right) + (1 - \theta) U^1 \left( U^1_{yy} U^2_x - U^1_{y} U^2_{xy} \right)}{\theta \left( U^1_{xx} U^2 + U^1 U^2_x \right) + (1 - \theta) \left( U^1_{x} U^2_x + U^1 U^2_{xx} \right)}, \quad (30)$$

but $\partial U^1/\partial \phi < 0$ for $\phi$ close to $y^*$ if we assume $\theta < 1$. If $\theta = 1$ then $\partial U^1/\partial \phi > 0$, in which case we cannot get cycles. The same is true for proportional bargaining. Although Nash bargaining with $\theta < 1$ implies similar results to price taking, as discussed in the Introduction, the economic forces are different.
Appendix B: Proofs

Proof of Proposition 1: Define $T(\phi) = f(\phi) - \phi$. Our assumptions imply $T'(0) > 0$. Also, $T'(\phi) = \beta - 1 < 0$ for $\phi > y^*$. By the continuity and monotonicity of $T(\phi)$ for $\phi > y^*$, it is easy to see the following: if $T(y^*) \geq 0$ there exists $\phi^* > y^*$ such that $T(\phi^*) = 0$; and if $T(y^*) < 0$ there exists $\phi^*$ in $(0, y^*)$ such that $T(\phi^*) = 0$. In the latter case, there is no stationary equilibrium in which $\phi^* > y^*$, because $T(\phi)$ is strictly decreasing for $\phi > y^*$. ■

Proof of Proposition 2: Because $f(\phi_t)$ is continuous, $\phi_{t-1}$ covers the interval $[0, \tilde{\phi}]$ for $\phi_t \in [0, \phi^*]$. Since there is a unique positive steady state, $f(\phi_t) > \phi_t$ for $\phi_t \in (0, \phi^*)$ and $f(\phi_t) < \phi_t$ for $\phi_t \in (\phi^*, \infty)$. That is $\phi_{t-1} > \phi_t$ for $\phi_t \in (0, \phi^*)$ and $\phi_{t-1} < \phi_t$ for $\phi_t \in (\phi^*, \infty)$. Given $\phi_0 < \tilde{\phi}$, there is a $\phi_1$ such that $\phi_1 \in (0, \phi^*)$ and $\phi_1 < \phi_0$, which implies a $\phi_2 \in (0, \phi^*)$ with $\phi_2 < \phi_1$, and so on. This sequence $\{\phi_t\}_{t=0}^\infty$ converges to 0. ■

Proof of Proposition 3: Let $f^2(\phi) = f \circ f(\phi)$. Because $\phi^*$ is the unique positive steady state $f(\phi) > \phi$ for $\phi < \phi^*$ and $f(\phi) < \phi$ for $\phi > \phi^*$. Because $f(\phi)$ is linearly increasing for $\phi > y^*$, there exists a $\tilde{\phi} > y^*$ such that $f(\tilde{\phi}) > y^*$. By the uniqueness of the positive stationary equilibrium, $f^2(\tilde{\phi}) < f(\tilde{\phi}) < \tilde{\phi}$. The slope of $f^2(\phi)$ is

$$\frac{df^2(\phi)}{d\phi} = f'(f(\phi)) f'(\phi) = f'(\phi) [f'(\phi)] = [f'(\phi)]^2 > 1.$$  

The last inequality uses $f'(\phi) < -1$. Similarly, $df^2(0)/d\phi > 0$. By continuity, $f^2$ must cross the 45° line in $(0, \phi^*)$. Because $f^2$ lies below the diagonal at $\tilde{\phi}$, it crosses it at least once in $(\phi^*, \tilde{\phi})$. So there are fixed points $\phi^1$ and $\phi^2$ such that $0 < \phi^1 < \phi^* < \phi^2$ for $f^2(\phi)$. ■

Proof of Proposition 4: Let $\phi^1, \phi^2, \ldots, \phi^n$ be the periodic points of a $n$-cycle. We prove the proposition in two steps.

Step 1: At least one periodic point is less than $\phi^*$. Suppose by way of contradiction that $\phi^j > \phi^*$ for all $j = 1, 2, \ldots, n$. By the definition of a $n$-period cycle, $\phi^1 = f(\phi^n) < \phi^n$, where the inequality follows from the fact that $f(\phi) < \phi$ for $\phi > \phi^*$ by the uniqueness of the positive steady state. Repeat the procedure starting from $\phi^n$ to get a contradiction:

$$\phi^n = f(\phi^{n-1}) < \phi^{n-1} = f(\phi^{n-2}) < \phi^{n-2} \ldots < \phi^1.$$  

Step 2: There does not exist a cycle if $\phi^* > y^*$. Suppose by way of contradiction that
there is a cycle and \( \phi^s > y^* \). By step 1, there exists at least one periodic point larger than \( \phi^s \). Let \( \phi^1 > \phi^s \). Then \( \phi^2 > \phi^s \) because \( \phi^2 = f(\phi^1) > f(\phi^* ) = \phi^s \), given \( f \) is strictly increasing for \( \phi > y^* \). Repeat the procedure to get \( \phi^i > \phi^s, i = 1, ..., n \), which contradicts step 1. So if there exists a cycle, \( \phi \) must be binding at some point. ■

**Proof of Proposition 5:** Since \( f'(\phi^s) < 0 \), there is an interval \([\phi^s - \varepsilon_1, \phi^s + \varepsilon_2]\), with \( \varepsilon_1, \varepsilon_2 > 0 \), such that \( f(\phi^1) > f(\phi^2) \) for \( \phi_1 \in [\phi^s - \varepsilon_1, \phi^s], \phi_2 \in (\phi^s, \phi^s + \varepsilon_2] \). By definition \((\phi_1, \phi_2)\) is a proper sunspot equilibrium if there exists \((\sigma_1, \sigma_2), \sigma_1, \sigma_2 < 1\), such that

\[
\begin{align*}
\phi_1 &= \sigma_1 f(\phi_1) + (1 - \sigma_1) f(\phi_2) \\
\phi_2 &= (1 - \sigma_2) f(\phi_1) + \sigma_2 f(\phi_2).
\end{align*}
\]

Because \( \phi_1 \) and \( \phi_2 \) are weighted averages of \( f(\phi_1) \) and \( f(\phi_2) \), where \( f(\phi_1) > \phi_1 \) and \( f(\phi_2) < \phi_2 \), by the uniqueness of the positive steady state, necessary and sufficient conditions for (31) and (32) are

\[
f(\phi_2) < \phi_1 < f(\phi_1) \quad \text{and} \quad f(\phi_2) < \phi_2 < f(\phi_1).
\]

Now, because \( \phi_1 < \phi_2 \), we can reduce this to

\[
\phi_2 < f(\phi_1) \quad \text{and} \quad \phi_1 > f(\phi_2).
\]

Expanding \( f(\phi_1) \) and \( f(\phi_2) \) around \( \phi^s \) and using \( f(\phi^s) = \phi^s \), the above inequalities are equivalent to

\[
\frac{\phi_2 - \phi^s}{\phi^s - \phi_1} < -f'(\phi^s) < \frac{\phi^s - \phi_1}{\phi_2 - \phi^s}.
\]

Because \(-f'(\phi^s) > 1\), \( \frac{\phi_2 - \phi^s}{\phi^s - \phi_1} < -f'(\phi^s) \) is redundant if \(-f'(\phi^s) < \frac{\phi^s - \phi_1}{\phi_2 - \phi^s}\). Now we have two unknowns \((\phi_1, \phi_2)\) and only one inequality \(-f'(\phi^s) < \frac{\phi^s - \phi_1}{\phi_2 - \phi^s}\) to solve. It is straightforward that multiple solutions exist on \([\phi^s - \varepsilon_1, \phi^s + \varepsilon_2]\). To show \( \sigma_1 + \sigma_2 < 1 \), rewrite (31)-(32) as

\[
\sigma_1 + \sigma_2 = \frac{\phi_1 - f(\phi_2) - \phi_2 + f(\phi_1)}{f(\phi_1) - f(\phi_2)} = \frac{\phi_1 - \phi_2}{f(\phi_1) - f(\phi_2)} + 1 < 1,
\]

because \( \frac{\phi_1 - \phi_2}{f(\phi_1) - f(\phi_2)} \) is negative. ■

**Proof of Proposition 6:** If \( \phi = y^* \), equilibrium is on the contract curve and \(-U^1_x/U^1_y = -U^2_x/U^2_y = y/x \). A calculation implies

\[
\frac{\partial U^1[h (\phi ), \phi]}{\partial \phi} \bigg|_{\phi = y^*} = \frac{U^1_y}{x} \left[ x^2 U^2_{xx} + 2xy U^2_{xy} + y^2 U^2_{yy} \right].
\]

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The term outside the brackets is negative. The term in brackets is positive as long as $y$ is normal for type 2. Similarly, with Nash bargaining,

$$
\frac{\partial U^1[h(\phi), \phi]}{\partial \phi} \bigg|_{\phi \rightarrow y_-^*} = \frac{\theta U^2 U^1_y \left( U^1_{xx} - \frac{U^1_1}{U^1_y} U^1_{xy} \right) + (1 - \theta) U^1 U^1_y \left( U^2_{xx} - \frac{U^2_1}{U^2_y} U^2_{xy} \right)}{\theta (U^1_{xx} U^2 + U^1_{x} U^2_x) + (1 - \theta) (U^1_{x} U^2 + U^1_{xx})}.
$$

Assuming $\theta < 1$, the denominator is negative, and the numerator is positive if $y$ is normal for both types.

**Proof of Corollary 1:** To construct cycles, it suffices to show we can choose $\beta$ and $\pi/\lambda$ such that $\phi^*$ is close to $y^*$ and $f'(\phi^*) < -1$. A steady state exists at $\phi = y^*$ if $y^* = (\beta \pi/\lambda) U^1 \left( x^*_-, y^*_+ \right) + \beta y^*$, or

$$
\pi \lambda = \frac{1 - \beta}{\beta} U^1 \big[ h(\phi), \phi \big] \bigg|_{\phi \rightarrow y_-^*}
$$

(35)

The derivative of $f(\phi)$ at $\phi = y^*$ is $f'(\phi) = (\beta \pi/\lambda) \partial U^1 \big[ h(\phi), \phi \big] / \partial \phi \bigg|_{\phi \rightarrow y_-^*} + \beta$. Hence, for $f'(\phi) < -1$ we need

$$
\frac{\beta \pi}{\lambda} \frac{\partial U^1 \big[ h(\phi), \phi \big]}{\partial \phi} \bigg|_{\phi \rightarrow y_-^*} + \beta < -1.
$$

(36)

Combining (35) and (36), after some algebra, we get

$$
\beta < \frac{-1 - \frac{\phi}{U^1 \big[ h(\phi), \phi \big]} \frac{\partial U^1 \big[ h(\phi), \phi \big]}{\partial \phi} \bigg|_{\phi \rightarrow y_-^*}}{1 - \frac{\phi}{U^1 \big[ h(\phi), \phi \big]} \frac{\partial U^1 \big[ h(\phi), \phi \big]}{\partial \phi} \bigg|_{\phi \rightarrow y_-^*}}
$$

(37)

If the elasticity condition holds, we can pick $\beta$ and $\pi/\lambda$ to satisfy (37).

**Proof of Corollary 2:** Given these utility functions, unconstrained and constrained equilibria satisfy $(x, y) = (1, 1)$ and $(x, y) = (\phi^{1/1+\gamma}, \phi)$. The unique positive steady state solves

$$
\phi - \frac{\beta}{1+\gamma} = (1 - \alpha) \left( 1 + \frac{1 - \beta \lambda}{\beta} \right)
$$

The steady state is constrained if and only if $\pi/\lambda < (1 - \alpha) (1 - \beta) / \alpha \beta$. A calculation implies $f'(\phi^*) < -1$ if and only if $\pi/\lambda > 1 + [(1 + \gamma) + (1 - \alpha)] / \beta (\alpha + \gamma)$. For these to hold simultaneously we need $\beta < \left[ \gamma (1 - \alpha) - \alpha (1 + \gamma) \right] / (\alpha + \gamma)$, with $\beta > 0$ as $(1 - \alpha) / \alpha > (1 + \gamma) / \gamma$.  

References


Figure 1a. Steady state with $\phi^s > y^*$

Figure 1b. Steady state with $\phi^s < y^*$

Figure 2a. Nonstationary equilibria

Figure 2b. Nonstationary equilibria
Figure 3a. Cycle with $\phi^1 < \phi^2 < y^*$

Figure 3b. Cycle with $\phi^1 < y^* < \phi^2$

Figure 4a. A three-period cycle

Figure 4b. Chaotic dynamics