Stochastic Discount Factor Models and the Equity Premium Puzzle

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Abstract

One view of the equity premium puzzle is that in the standard asset-pricing model with time-separable preferences, the volatility of the stochastic discount factor, for plausible values of risk aversion, is too low to be consistent with consumption and asset return data. We adopt this characterization of the puzzle, due to Hansen and Jagannathan (1991), and establish two results: (i) resolutions of the puzzle based on complete frictionless markets and non-separabilities in preferences are very sensitive to small changes in the consumption data, and (ii) models with frictions avoid this sensitivity problem. Using quarterly data from 1947-97, we calibrate a state non-separable model and a time non-separable model to satisfy the Hansen-Jagannathan volatility bound and show that the two resolutions are not robust. We support our argument via a bootstrap experiment where the models almost always violate the bound. These violations are primarily due to the fact that small changes in consumption growth moments imply changes in the mean of the stochastic discount factor, which render the volatility of the stochastic discount factor to be too low relative to the bound. Asset-pricing models with frictions, however, are much more successful in the bootstrap experiment relative to the case without frictions.

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I. **Introduction**

Mehra and Prescott (1985) pointed out that the historical size of the equity premium in the U.S. is too high to be explained by an intertemporal asset-pricing model, such as that of Lucas (1978). In a model with complete frictionless markets and CRRA time-separable preferences, they showed that reasonable values of risk aversion do not reproduce the observed equity premium. Resolutions of this *equity premium puzzle* have followed two distinct paths. One approach was to retain the complete frictionless markets framework, but abandon the separability assumptions in the preferences. Prominent examples of this approach are Weil (1989) and Epstein and Zin (1991), who use state non-separable preferences, and Constantinides (1990), who uses time non-separable preferences. The second approach to resolving the equity premium puzzle abandons the complete frictionless markets framework. Examples of this approach include Aiyagari and Gertler (1991), Lucas (1994) and Heaton and Lucas (1996), who consider uninsured risk and transactions costs.

One way to view the equity premium puzzle is that the standard intertemporal asset-pricing model together with consumption data does not deliver a sufficiently volatile stochastic discount factor (or, synonymously, intertemporal marginal rate of substitution) to be consistent with asset return data. This characterization is due to Hansen and Jagannathan (1991). They developed a lower bound on the volatility of the intertemporal marginal rate of substitution (IMRS) and showed that the model with reasonable parameters for time separable preferences violated the lower bound. Specifically, they showed that the volatility of each stochastic discount factor that satisfies the representative consumer’s Euler equation must exceed a lower bound (the HJ bound). Using consumption and asset return data for the U.S. they demonstrated that the volatility of the IMRS was too small for plausible values of risk aversion.

In this paper, we take the Hansen-Jagannathan view of the equity premium puzzle and establish two results: (i) resolutions of the puzzle based on complete frictionless markets are very sensitive to small changes in the consumption data, and (ii) models with frictions avoid this sensitivity problem.

To demonstrate the first result, we use quarterly data on equity returns, Tbill returns and consumption growth from 1947-97 and calibrate the two models with non-separabilities to satisfy the HJ bound. We then conduct a bootstrap experiment and show that the two models violate the HJ bound in almost all samples. Specifically, we draw time series samples from the joint ‘empirical’ distribution of the IMRS, equity return and T-bill return for the period 1947-97 and show that the models routinely violate the

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1 Mehra and Prescott (1985) suggest this approach in their concluding remarks -- “Perhaps introducing some features that make certain types of intertemporal trades among agents infeasible will resolve the puzzle.” p.159.
2 For a recent description of this view, see Ljungqvist and Sargent (2000), pp. 263-271.
3 The HJ bound is a function of the mean of each risky return, the contemporaneous covariance matrix of returns and the mean price of a unit payoff riskless asset. In the absence of the riskless asset, the HJ bound is calculated using the mean IMRS.
bound. These violations are primarily due to the fact that small changes in consumption growth moments imply changes in the mean IMRS which render the volatility of the IMRS to be too low relative to the HJ bound.

Our evidence for the first result strengthens Chapman’s (2001) argument against Constantinides’s habit formation model. Chapman argues that the behavior of consumption growth pre 1948 is very different from the behavior post 1948 and that the habit formation model cannot resolve the equity premium puzzle in the post-1948 data. (See also Golob (1994) for evidence on changes in the consumption growth process.) In this paper we work only with post-war data and show that the habit formation model calibrated to satisfy the HJ bound in the 1947-97 data violates it in other post-war subsamples.⁴ We demonstrate this for the state non-separable model as well.

To establish the second result regarding frictions, we use the volatility bound developed by He and Modest (1995) and Luttmer (1996). Their bound is an extension of the HJ bound to an environment with frictions such as short-sales constraints. We show that the models are much more successful in the bootstrap experiment relative to the case without frictions. Frictions in these models do not directly affect the volatility of the stochastic discount factor. Instead, they lower the HJ bound implied by the asset return data and, hence, enable the model to satisfy the bound. Moreover, the frictions flatten out parts of the HJ bound frontier in ways that help avoid the sensitivity problem noted earlier -- the lower bound on volatility is roughly constant for a wide (relevant) range of means of stochastic discount factors.

II. Asset Returns, Consumption and the Equity Premium

For preference-based asset pricing models, Hansen and Jagannathan (1991) deliver a lower bound on the volatility of the IMRS. The bound can be calculated from the asset return data, independently of the model’s IMRS. For an asset-pricing model to be valid, the volatility of the model's IMRS must exceed this bound. The presentation of the bound here is brief. The details of the HJ bound are included in Appendix A purely for completeness.

Let \( R \) denote the \( n \times 1 \) (gross) return vector of risky assets and let \( v \) denote the mean price of the unit payoff fictitious riskless asset. Consider two stochastic discount factors \( m \) and \( m_v \) that price the assets according to

\[
\tau = ERm = ERm_v,
\]

⁴ In related work, Otrok, Ravikumar and Whiteman (2001) show that the equity premium in habit formation model is generated primarily by the (negligible) amount of high frequency volatility in the consumption data. Hence, the equity premium implication of the habit formation model is extremely sensitive to small changes in the high frequency component of consumption.
where \( \mathbf{1} \) is an \( n \times 1 \) vector of ones. The above equation is the unconditional version of the standard Euler condition equating the expected marginal cost and marginal benefit of delaying consumption one period. (Note that if there was indeed a riskless asset in the economy, then \( v = \mathbb{E} m \).) For all \( m \)'s such that \( \mathbb{E} m = \mathbb{E} m_{\mathbf{1}} = v \), Hansen and Jagannathan (1991) show that the lower bound on the \( \text{var}(m) \) is given by

\[
\text{var}(m_{\mathbf{1}}) = (1 - v \mathbb{E} R)^{-1} \Omega^{-1} (1 - v \mathbb{E} R),
\]

where \( \Omega \) is the covariance matrix of risky-asset returns. The lower bound, \( \text{var}(m_{\mathbf{1}}) \), is a function of the arbitrarily picked \( v \). Thus, by picking different \( v \)'s we generate a lower bound frontier. A necessary condition for a stochastic discount factor with mean \( v \) to be consistent with asset return data is that its variance exceeds the bound in (1).

II.1 An Asset Pricing Puzzle

We calculate the HJ bound using quarterly equity (S&P 500) and Treasury bill returns from 1947-1997 and equation (1).\(^5\) We also calculate the volatility of the representative agent's IMRS for CRRA time-separable preferences used by Mehra and Prescott (1985). The preferences are described by:

\[
U_o = \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\sigma}}{1-\sigma}, \quad \sigma > 0,
\]

where \( \mathbb{E}_0 \) denotes conditional expectation given information at time 0, \( c_t \) denotes consumption at time \( t \), \( \beta \in (0,1) \) is the discount factor, and \( \sigma \) is the measure of relative risk aversion. (The \( \sigma = 1 \) case will be interpreted as logarithmic.) The IMRS for these preferences is given by:

\[
m_{t+1} = \beta \left( \frac{c_{t+1}}{c_t} \right)^{\sigma}.
\]

Figure 1 below plots the HJ bound and the volatility of the above IMRS using quarterly consumption data from 1947-1997. In calculating the IMRS volatility, we set \( \beta = 0.99 \) and let \( \sigma \) vary from 225 to 263. The HJ bound is the solid curve; the IMRS volatility is represented by the squares.

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\(^5\) The data are all in real terms. Equity returns were calculated using the S&P 500 stock price and dividends from the Citibase dataset. Consumption is measured by per capita consumption of nondurables and services.
The time-separable model generates enough volatility to satisfy the bound when $\sigma = 263$. Thus, the model violates the bound unless the coefficient of risk aversion is implausibly large. In other words, for plausible values of risk aversion, the volatility of the stochastic discount factor implied by the model is too low relative to that implied by the asset return data.

II.2 Resolutions of the Puzzle

We concentrate on two resolutions of the puzzle in this sub-section. The resolutions are based on relaxing separability in the utility function, in one case state separability and in the other time separability. Both add just 1 parameter to the Mehra-Prescott model and both increase the volatility in the stochastic discount factor.

Epstein and Zin (1991) and Weil (1989) generalized the time-separable preferences to allow for an independent parameterization of attitudes towards risk and intertemporal substitution. Following Weil (1989), we assume that these state-non-separable preferences are given by:

$$V_t = U[c_t, E_t V_{t+1}]$$

where

$$U[c, V] = \left\{ (1-\beta)c^{1-\rho} + \beta [1 + (1 - \beta)(1 - \sigma)V] \right\} \beta^{(1-\sigma)/(1-\rho)} - 1$$

The elasticity of intertemporal substitution is $1/\rho$ and $\sigma$ is the coefficient of relative risk aversion.

The IMRS for these preferences is:
where $R_{t+1}$ is the return on the market portfolio.

Constantinides (1990) models consumers as habitual, in that levels of consumption in adjacent periods are complementary. That is, the time-non-separable preferences of consumers (in a discrete-time version of Constantinides, 1990) are given by:

$$U_0 = E_o \sum_{t=0}^{\infty} \beta^t \frac{[(1 + \delta(L))c_t]^{1-\sigma}}{1-\sigma},$$

where $\delta(L)$ is a polynomial in the lag operator $L$. When the lag coefficients are all negative, the preferences exhibit habit-persistence. When the coefficients in $\delta(L)$ are zero, the preferences are time-separable. Here we work with the popular one-lag habit case with $\delta(L) = \delta L$, where $\delta < 0$.

The representative agent's IMRS is given by:

$$m_{t+1} = \beta \left( c_{t+1} + \delta c_t \right)^{-\sigma} + \beta E_{t+1} \left( (c_{t+2} + \delta c_{t+1})^{-\sigma} - \delta E_{t+1} \left( c_{t+1} + \delta c_t \right)^{-\sigma} \right).$$

Figure 2 below plots the bound and the IMRS volatilities for the two models. Again, the HJ bound is the solid curve; the squares represent the IMRS volatility. For state-non-separable preferences in panel a, the parameters are $\beta = 0.99$, $\rho = 3.15$ and $\sigma$ ranging from 15 to 16.4. For time-non-separable preferences in panel b, the parameters are $\beta = 0.96$, $\delta = -0.72$, and $\sigma$ ranging from 3.0 to 3.3.

**Figure 2: Resolutions of the Asset Pricing Puzzle, 1947-1997**

- **a: Epstein-Zin**
- **b: Habit Formation**
These are resolutions in the sense that the models deliver sufficient IMRS volatility with a $\sigma$ that is not excessively large, and an elasticity of intertemporal substitution that is not excessively small. (Note that $\sigma$ is not the coefficient of risk aversion in the habit model, though it is proportional to various measures of risk aversion. See Boldrin, Christiano, and Fisher, 1997.) It is difficult to say whether the habit parameter is ‘unreasonable’ since there is very little micro evidence on this parameter. In the next section, we show that the resolutions are extremely sensitive to changes in the underlying consumption process.

III. Problems with the Resolutions

Consider a subsample, 1957-1987, of the overall sample. As we can see in Table 1 the moments are very similar for consumption growth but differ for returns in the two sample periods. (In fact, mean consumption growth is the same for both samples up to the third decimal point.) The correlation between T-bill and equity returns is 0.08 for the period 1947-97 and 0.13 for the period 1957-87.

Using the model parameters from the previous section together with the asset return and consumption data from 1957-1987, we calculate the HJ bound and IMRS volatility. (Recall that the period was 1947-1997 in the previous section.) As we see in Table 2, though both models satisfy the bound by construction for 1947-1997, they violate it in dramatic fashion for the period 1957-87.

An examination of Table 1 suggests initially that the failure of the two models in the 1957-87 subsample could be due to the changes in asset return moments, since there is little change in the consumption growth data. The HJ bound is a function of the asset return moments, so the changes in asset returns could have affected it and the failure of the two models could therefore be due to movement of the

### Table 1

<table>
<thead>
<tr>
<th>Cons. Growth</th>
<th>Equity Returns</th>
<th>T-Bill Returns</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>mean</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1947-1997</td>
<td>0.45</td>
<td>2.06</td>
</tr>
<tr>
<td>1957-1987</td>
<td>0.45</td>
<td>1.38</td>
</tr>
<tr>
<td><strong>std dev</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1947-1997</td>
<td>0.55</td>
<td>5.80</td>
</tr>
<tr>
<td>1957-1987</td>
<td>0.52</td>
<td>6.08</td>
</tr>
</tbody>
</table>

### Table 2

<table>
<thead>
<tr>
<th>IMRS Mean</th>
<th>IMRS Std Dev</th>
<th>HJ Bound</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Epstein-Zin</strong></td>
<td><strong>1947-97</strong></td>
<td><strong>1957-87</strong></td>
</tr>
<tr>
<td>0.998</td>
<td>0.956</td>
<td>0.325</td>
</tr>
<tr>
<td><strong>Habit</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.997</td>
<td>0.981</td>
<td>0.311</td>
</tr>
</tbody>
</table>

An examination of Table 1 suggests initially that the failure of the two models in the 1957-87 subsample could be due to the changes in asset return moments, since there is little change in the consumption growth data. The HJ bound is a function of the asset return moments, so the changes in asset returns could have affected it and the failure of the two models could therefore be due to movement of the
“target”—the HJ bound itself. In Figure 3 we plot the HJ bound frontier for the two sample periods and they are almost indistinguishable for a wide range of means. Thus, the failure of the two models is not due to the changes in asset return moments and the resulting changes in the HJ bound frontiers.

Figure 3: The HJ Bound, 1947-1997 and 1957-1987

Another possibility is that the volatility of the stochastic discount factors is “substantially” different despite small changes in the consumption growth moments. However, the IMRS volatility turns out to be remarkably stable: in the model with state non-separable preferences, in moving from the whole sample to the subsample, the standard deviation of the IMRS only increases from 0.325 to 0.340. Had the HJ bound remained roughly the same, as suggested by Figure 3, the Epstein-Zin model should not violate the HJ bound.

We are thus led to a third possibility: the failure of the models in the 1957-87 subsample must be due to the change in the mean IMRS and the resulting impact on the HJ bound. A favorable clue for pursuing this comes from Table 2. The mean IMRS for the state-non-separable preferences fell from 0.998 to 0.956, while the HJ bound increased by more than a factor of 20, from 0.32 to 7.09; for time-non-separable preferences, the mean IMRS fell from 0.997 to 0.981 and the HJ bound increased by a factor of 9, from 0.30 to 2.76. To reconcile this with Figure 3, note that the figure not only demonstrates that changes in the asset return moments have negligible effects on the HJ bound at any given mean IMRS, but also demonstrates that the bounds are very different for small differences in the mean IMRS. That is, the sides of the “V” are very steep. Thus, even though the HJ bound frontiers look identical in Figure 3, a small change in the mean IMRS implies a “large” change in the HJ bound. The failure of the models in the 1957-87 subsample must therefore be due to the fact that we are evaluating the HJ bound frontiers at different mean IMRSs.
To summarize, the HJ bound frontier seems to be unaffected by changes in asset return moments, but the value of the HJ bound seems to be sensitive to the mean IMRS. Therefore, the success of the two models seems to hinge critically on the mean IMRS since the lower bound on the volatility of the stochastic discount factor varies dramatically with the mean IMRS.

Several remarks are in order at this stage. First, in the context of models with time-separable preferences and habit formation preferences, Cecchetti, Lam and Mark (CLM, 1994) make the related but distinct point that in statistical tests of the ‘distance’ between IMRS volatility and the HJ bound, much of the uncertainty in the distance is due to uncertainty in estimating the mean of the IMRS. Furthermore, they state that the uncertainty in the mean IMRS is due to considerable uncertainty in consumption growth moments. However, our analysis so far suggests that even though the first two moments of consumption growth are virtually the same in both samples (Table 1), the HJ bounds are quite different (Table 2). We argue that the changes in the consumption process are small, but preference-based complete-frictionless-markets resolutions are extremely sensitive to minor changes in the consumption process. Our argument holds not only for the habit formation model considered by CLM but also for the state non-separable model. Burnside (1994) studies the small sample properties of statistical measures of the HJ bound (see also Otrok, Ravikumar and Whiteman 2002). He extends the literature on statistical tests of the HJ distance measure by examining the small-sample distributions of the test statistics. He finds that asymptotic theory is not a good approximation to the finite sample distributions. Burnside argues that part of the poor performance of the asymptotic statistics (essentially, over rejecting true models) can be attributed to variations in the mean of the IMRS in his simulations of ‘true’ models. While our point is related, our study is of the sensitivity of resolutions of asset pricing puzzles to changes in the observed consumption data while Burnside studies the properties of a test statistic in a controlled experiment where the asset pricing model is true and the econometrician specifies one forcing process for consumption, which is known by both the econometrician and agent in the model.

Second, we are viewing the preference parameters as fixed and examining how the model responds to changes in consumption growth and asset return moments. Of course, it will generally be possible to find a set of parameter values specific to the subsample period that will enable the model to achieve the HJ bound for that subsample. Yet introducing as many free parameters as subsamples renders the model’s “explanation” of the data largely meaningless. Still, it might be argued that such calculations are of interest if the parameter values do not vary “too much”. However, to achieve the bound in the 1957-1987 sample holding other parameters fixed, $\sigma$ must be increased by 20-40% in the two models. In the Epstein-Zin model, where $\sigma$ is the coefficient of relative risk aversion, the increase is from 16.4 to 23.6, a very large value. Interpreting the increase in $\sigma$ in the habit model is more subtle since it is not the coefficient of relative risk aversion. For instance, the increase in $\sigma$ from 3.3 to 4 causes the implied equity premium in the model to rise
from 6% to 12% (annual). (The actual equity premium in the 1957-87 subsample is 3.5%.) These calculations suggest that in order to achieve the HJ bound in the subsample, fundamental parameters would need to be varied “too much”.

Third, the failures of the two models are not specific to the 1957-87 subsample. Consider a sequence of 80-quarter rolling subsamples, 1947-67, 1950-70, etc. Table 3 below evaluates the two models by documenting the volatility of the stochastic discount factor and the HJ bound for each rolling subsample. Again, the model parameters are the same as in Section II. (To facilitate comparison, we have repeated the information for the overall sample, 1947-97.)

<table>
<thead>
<tr>
<th>Sample</th>
<th>Epstein-Zin</th>
<th>Habit Formation</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>IMRS Mean</td>
<td>HJ Bound Mean</td>
</tr>
<tr>
<td>1947-97</td>
<td>0.998</td>
<td>0.316</td>
</tr>
<tr>
<td>1947-67</td>
<td>0.999</td>
<td>0.518</td>
</tr>
<tr>
<td>1950-70</td>
<td>0.970</td>
<td>5.319</td>
</tr>
<tr>
<td>1953-73</td>
<td>0.973</td>
<td>9.445</td>
</tr>
<tr>
<td>1956-76</td>
<td>0.930</td>
<td>19.381</td>
</tr>
<tr>
<td>1959-79</td>
<td>0.904</td>
<td>24.903</td>
</tr>
<tr>
<td>1962-82</td>
<td>0.907</td>
<td>16.849</td>
</tr>
<tr>
<td>1965-85</td>
<td>0.921</td>
<td>11.425</td>
</tr>
<tr>
<td>1968-88</td>
<td>0.959</td>
<td>5.394</td>
</tr>
<tr>
<td>1971-91</td>
<td>0.983</td>
<td>1.910</td>
</tr>
<tr>
<td>1974-94</td>
<td>1.005</td>
<td>1.436</td>
</tr>
<tr>
<td>1977-97</td>
<td>1.036</td>
<td>7.318</td>
</tr>
</tbody>
</table>

It is clear that in all subsamples, the models violate the HJ bound. Furthermore, there is hardly any change in the volatility of the IMRS across the subsamples relative to the change in the HJ bound. For the Epstein-Zin model, the range of the mean IMRS across the subsamples is roughly 0.91 to 1.04 while the HJ bound ranges from 0.52 to 24.9; for the habit formation model, the mean IMRS ranges from 0.976 to 1.03 while the HJ bound ranges from 1.86 to 5.1. As in the previous section, the changes in the mean IMRS significantly affect the HJ bound.

In the next section, we examine whether the results in this section are robust by conducting a bootstrap experiment with the entire sample, 1947-97.

IV. A Bootstrap Experiment

In this section we investigate how the HJ bound and the volatility of the stochastic discount factor vary across artificial samples drawn from the entire data set, 1947-97. To do this we calculate time series for
the representative agent’s stochastic discount factor for the two asset-pricing models using actual consumption growth data and then use a bootstrap procedure to sample a vector of asset returns and an IMRS. We bootstrap the entire vector so that the observed correlation properties between the 2 returns and the IMRS are maintained in our experiment.  

The bootstrap procedure is as follows:

1) Use the structural parameters from Section II and observed consumption growth data to get time-series for the IMRSs of the two models.

2) Draw (with replacement) a time series of length 200 from the joint ‘empirical’ distribution of the IMRSs, equity returns and T-bill returns. That is for each period we draw a 3-tuple (IMRS, \( R_{equity} \), \( R_{T-bill} \)).

3) Calculate the mean and volatility of the IMRS.

4) Calculate the HJ bound using the time series for equity and T-bill returns at the mean IMRS.

5) Repeat steps 2-4 1000 times.

Figure 4 below is a scatter plot of the vertical distance between the HJ bound and IMRS volatility, calculated as the IMRS volatility minus the HJ bound, for each of the 1000 bootstrap simulations.

The striking feature of this picture is that the distance is almost always negative, implying that the models miss the bound in most simulations. In fact, the time-non-separable model misses in 97.5% of the simulations and the state-non-separable model misses in 97.3% of the simulations.

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6 Cochrane and Hansen (1992) show that the correlation between asset returns and the candidate discount factor affects how high the bound needs to be; when the candidate is less correlated with return data it must be more volatile to satisfy the bound.
Figure 5 below plots the volatility of the IMRSs of the two models. In both models, the volatility is not too disperse in the following sense: modest changes in the mean IMRS are not accompanied by large changes in the volatility of the IMRS.

**Figure 5: IMRS Volatility**

<table>
<thead>
<tr>
<th>a: Epstein-Zin</th>
<th>b: Habit Formation</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="image1" alt="Graph a: Epstein-Zin" /></td>
<td><img src="image2" alt="Graph b: Habit Formation" /></td>
</tr>
</tbody>
</table>

Note also that the distance in Figure 4 is an order of magnitude larger than the IMRS volatility, indicating that the violations are due to large changes in the HJ bound. This contrast is illustrated in Figure 6. The HJ bound in Figure 6 is not the HJ bound frontier, but is an envelope connecting the lowest HJ bounds in the bootstrap experiment. That is, each bootstrap sample yields a mean IMRS and an associated HJ bound; potentially, there are samples with the same mean IMRS, but a different HJ bound; the envelope picks the lowest HJ bound for each mean IMRS. The figure illustrates that the sample variations in the IMRS mean is vastly more important in generating violations of the bound than the variation in IMRS volatility.
In Figure 7, we provide a scatter plot of the consumption growth moments and return moments from the bootstrap experiment. It is clear from these pictures that there is substantially more variability in the return moments than the consumption growth moments. In fact, the variability in equity returns is an order of magnitude greater than the variability in consumption growth moments! Yet, the model’s success depends critically on the mean IMRS, which is a function just of the consumption growth moments.
We also considered an alternative bootstrapping method. We fit a VAR to the returns and IMRS, and then bootstrapped from the VAR residuals. This bootstrap, by construction, matched more properties of the data, such as autocovariances and cross-autocovariances. The results from this experiment were nearly identical to those reported above.

V. Asset-Pricing with Frictions

One approach to ensure success of the asset-pricing models is to develop environments in which the HJ bound is not sensitive to changes in the mean IMRS. While it is not clear how one goes about constructing such environments, it turns out that the models developed by He and Modest (1995) and Luttmer (1996) do yield the necessary lack of sensitivity. They find that frictions such as short-sales constraints, borrowing constraints, transactions costs, and solvency constraints can resolve the asset-pricing puzzles better than the complete markets consumption-based asset-pricing models. In this section, we study economies with short-sale constraints and provide results on exercises similar those in the previous sections.

In the presence of short sales constraints, the representative agent’s Euler equation is an inequality:

\[ \text{ERm} \leq \iota. \]

For assets with no short sale constraints, the restriction holds with equality. Let us rewrite the inequality as

\[ \text{ERm} = \chi, \; \chi \leq \iota, \]

where \( \chi \) is a vector of unknown parameters. Given \( \chi \) and \( \nu \), He and Modest (1995) show that the HJ bound for this economy is

\[ \text{var}(m_v) = (\chi - \nu \text{ER})\Omega^{-1}(\chi - \nu \text{ER}). \]

Frictions have also proven to be useful in addressing other empirical observations on individual consumptions, portfolio holdings and wealths. See Aiyagari (1993) for a call to develop asset-pricing models with frictions.
They restrict all the Lagrange multipliers on the short-sale constraints to be the same. Since they are interested in constructing a lower bound, they choose the unknown $\chi$ to minimize $\text{var}(m_v)$.

Figure 8 illustrates that the imposition of frictions allows the standard time-separable stochastic discount factor model in equation (2) to satisfy the bound with lower levels of risk aversion. The level of risk aversion is now 59.8, rather than 263. Note that the HJ bound frontier is relatively flat (at least on one side) compared to that in Figure 1. Thus, there is indeed a possibility that small changes in the mean IMRS will not result in large changes in the HJ bound.

Figure 8: HJ Bound with Frictions and Time-Separable Preferences

An inspection of Figures 2 and 8 suggests that frictions have not altered the relevant regions of the HJ bound frontier where the IMRS volatilities for the non-separable models exceed the bound (near mean IMRSs exceeding 0.99). Hence, their parameters remain the same. (Recall that for state-non-separable preferences the parameters are $\beta = 0.99$, $\rho = 3.15$ and $\sigma = 16.4$; for time-non-separable preferences the parameters are $\beta = 0.96$, $\delta = -0.72$ and $\sigma = 3.3$.)

Now, consider the subsample 1957-87. Table 4 below illustrates that the asset-pricing models with frictions are successful in both sample periods.
Table 4

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Time-Separable</td>
<td>0.800</td>
<td>0.796</td>
<td>0.280</td>
<td>0.264</td>
<td>0.247</td>
<td>0.108</td>
</tr>
<tr>
<td>Epstein-Zin</td>
<td>0.998</td>
<td>0.956</td>
<td>0.325</td>
<td>0.340</td>
<td>0.315</td>
<td>0.129</td>
</tr>
<tr>
<td>Habit</td>
<td>0.997</td>
<td>0.981</td>
<td>0.311</td>
<td>0.249</td>
<td>0.307</td>
<td>0.133</td>
</tr>
</tbody>
</table>

Recall from Table 1 that the consumption growth moments are similar for the two sample periods, while the asset return moments are not. While the mean IMRSs remain the same as in Table 2, we see that the bound has fallen quite dramatically. Figure 9 shows how the bound has fallen in the 1957-1987 sample.

Figure 9: HJ Bound with Frictions – 1947-97 versus 1957-87

Table 5 illustrates a similar result via other subsamples. The model with time-separable preferences achieves the lower bound in 8 of 11 subsamples, and is very close in a 9th subsample. For Epstein-Zin preferences the model achieves the bound in 7 of the subsamples and is very close in 2 other subsamples. The habit model achieves the bound in 7 out of 11 subsamples. This stands in stark contrast to Table 3, where the Epstein-Zin and Habit models violated the bound in almost all the subsamples.
In Figure 10, we illustrate the results from the bootstrap experiment for the case with frictions. The distance to the bound is generally positive in the bootstrap experiment, indicating that the model satisfies the bound. The model with time-separable preferences violates the bound in 35.5% of the samples, the Epstein-Zin model violates the bound in 46% and the Habit model violates the bound in 52%.

**Table 5**

<table>
<thead>
<tr>
<th>sample</th>
<th>Time-Separable</th>
<th>Epstein-Zin</th>
<th>Habit Formation</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>IMRS Mean</td>
<td>Std Dev</td>
<td>HJ Bound</td>
</tr>
<tr>
<td>1947-97</td>
<td>0.800</td>
<td>0.280</td>
<td>0.247</td>
</tr>
<tr>
<td>1947-67</td>
<td>0.742</td>
<td>0.309</td>
<td>0.384</td>
</tr>
<tr>
<td>1950-70</td>
<td>0.711</td>
<td>0.250</td>
<td>0.314</td>
</tr>
<tr>
<td>1953-73</td>
<td>0.735</td>
<td>0.232</td>
<td>0.249</td>
</tr>
<tr>
<td>1956-76</td>
<td>0.780</td>
<td>0.258</td>
<td>0.096</td>
</tr>
<tr>
<td>1959-79</td>
<td>0.773</td>
<td>0.241</td>
<td>0.052</td>
</tr>
<tr>
<td>1962-82</td>
<td>0.822</td>
<td>0.282</td>
<td>0.002</td>
</tr>
<tr>
<td>1965-85</td>
<td>0.822</td>
<td>0.281</td>
<td>0.018</td>
</tr>
<tr>
<td>1968-88</td>
<td>0.833</td>
<td>0.270</td>
<td>0.058</td>
</tr>
<tr>
<td>1971-91</td>
<td>0.858</td>
<td>0.289</td>
<td>0.098</td>
</tr>
<tr>
<td>1974-94</td>
<td>0.847</td>
<td>0.270</td>
<td>0.157</td>
</tr>
<tr>
<td>1977-97</td>
<td>0.843</td>
<td>0.256</td>
<td>0.255</td>
</tr>
</tbody>
</table>

**Figure 10: Distance to the HJ Bound with Frictions**

a: Time-Separable

b: Epstein-Zin
Figure 11 below illustrates that the mean and volatility of the IMRS for the time-separable model with frictions is, in some sense, similar to those illustrated for the non-separable models in Figure 5. (The figures for the Epstein-Zin and Habit models are the same as in Figure 5, since the model parameters remain the same.) However, contrary to the case in Figure 5, the success of the asset-pricing models with frictions does not hinge critically on the mean IMRS. Note also that the volatility of the IMRS is roughly the same order of magnitude as the distance in Figure 10.

Another alternative to ensure success of the asset-pricing models is to retain the frictionless markets, but construct preferences that yield a roughly constant mean IMRS. Campbell and Cochrane (1999) constructed a functional form with slowly adjusting external habit that yields a risk free rate that is constant across states of a given consumption growth process. If the risk free rate in their model turns out to be roughly constant across different consumption growth processes (i.e., different consumption growth
moments), then to a first approximation their model may deliver a roughly constant mean IMRS. In Appendix B, we show that such an alternative does not fare better than the Epstein-Zin or Habit formation preferences.

VI. Conclusion

We have documented the sensitivity of estimates of the moments of an agent’s intertemporal marginal rate of substitution to small changes in the underlying consumption growth process. Specifically, small changes in moments of consumption growth, changes that are modest relative to sampling error in their estimates, lead to changes in the mean IMRS. Further, estimates of the lower bound mean-variance frontier for acceptable IMRSs do not vary much when taking into account the uncertainty in the first and second moments of historical returns, despite the fact that the return moments can vary dramatically. However, the changes in the mean IMRS do have a dramatic effect on the Hansen-Jagannathan volatility bound relative to the volatility of the IMRS. In other words, the HJ bound frontier, as a function of the mean IMRS, is very steep, while the volatility of the IMRS viewed as a function of the mean is quite flat. We have shown that complete-markets asset pricing models, parameterized using one historical sample on consumption data to satisfy the Hansen-Jagannathan bound, will violate it for many if not most other historical samples. Asset-pricing models with frictions do not share this sample sensitivity. In such models, while changes in consumption growth moments continue to affect the mean IMRS, the Hansen-Jagannathan bound is no longer as sensitive to the changes in the mean IMRS.
References


Appendix A: The Hansen-Jagannathan Bound

We seek a stochastic discount factor $m_v$ with mean $v$ that satisfies:

(A1) \[ \eta = \text{ER}m = \text{ER}m_v \]

and lives in a convenient space (specified below). For all stochastic discount factors such that $\text{Em} = \text{Em}_v = v$, we can rewrite (A1) as

(A2) \[ \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \text{ER}_v m = \text{ER}_v m_v, \]

where $R_v$ denote the return vector for the $n+1$ assets, with the first element equal to $1/v$. Then

(A3) \[ \text{ER}_v (m - m_v) = 0. \]

Let $P$ be the linear space given by \{ $R_v \lambda : \lambda \in \mathbb{R}^{n+1}$ \}, and suppose that $m_v$ is in $P$. (For example, an $m_v$ in $P$ satisfying (A2) is $m_v = R_v \lambda$, where $\lambda = [\text{ER}_v R_v']^{-1}$.) Define the "error" $\varepsilon$ such that

(A4) \[ m = m_v + \varepsilon. \]

Note that $\text{Em} = \text{Em}_v$ implies $E\varepsilon = 0$, and from (A3), $\text{ER}_v \varepsilon = 0$. Thus $m_v$ is the linear least squares projection of $m$ onto $P$. Then

\[
\text{var}(m) = \text{var}(m_v) + \text{var}(\varepsilon) + 2\text{cov}(m_v, \varepsilon) = \text{var}(m_v) + \text{var}(\varepsilon) + \text{Em}_v \varepsilon.
\]

Since $m_v$ is in $P$ and $\varepsilon$ is orthogonal to $P$, we must have $\text{Em}_v \varepsilon = 0$. Thus, we have

\[
\text{var}(m) = \text{var}(m_v) + \text{var}(\varepsilon) \geq \text{var}(m_v),
\]

meaning that a lower bound on the variance of $m$ is that of $m_v$.

To find this lower bound, note that since $m_v$ is a linear combination of the elements of $P$, it can be written in the form

(A5) \[ m_v = v + (R - \text{ER})'\beta, \]

for $\beta$ in $\mathbb{R}^n$. To obtain the vector $\beta$, subtract $v$ from both sides, multiply by $(R-\text{ER})$ and take expectations:

(A6) \[ \text{E}(R - \text{ER})(m_v - v) = \text{E}(R - \text{ER})(R - \text{ER})'\beta = \Omega \beta, \]

where $\Omega$ is the covariance matrix of risky-asset returns. A more convenient expression for the left-hand-side of (A6) can be derived as follows. First, from (A2), we have

\[ \text{ER}_v (m_v - v) = \begin{bmatrix} 1 & 1 \end{bmatrix}' - v \text{ER}_v . \]

Now subtract zero \[ (= \text{ER}_v \text{E}(m_v - v)) \] from the left-hand-side to obtain

\[ \text{ER}_v (m_v - v) - \text{ER}_v \text{E}(m_v - v) = \begin{bmatrix} 1 & 1 \end{bmatrix}' - v \text{ER}_v , \]

or

(A7) \[ \text{E}(R_v - \text{ER}_v)(m_v - v) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} - v \begin{bmatrix} v^{-1} \\ \text{ER}_v \end{bmatrix}. \]
Since the first row is just an identity corresponding to the fictitious asset, we can rewrite equation (A7) as
\[ E(R - ER)(m_v - v) = \nu - vER. \]
Therefore, we can solve (A6) for \( \beta \) as
\[ \beta = \Omega^{-1}(\nu - vER). \]
Thus,
\[
\text{var}(m_v) = E(m_v - v)'(m_v - v) \\
= \beta' E(R - ER)(m_v - v) \quad \text{(from equation (A5))} \\
= \beta' \Omega \beta \quad \text{(from equation (A6)).}
\]
Substituting for \( \beta \) from (A8), we can write
\[ \text{var}(m_v) = (\nu - vER)'\Omega^{-1}(\nu - vER). \]
Appendix B: External Habit and the Equity Premium Puzzle

In this appendix, we use the preferences specified by Campbell and Cochrane (1999). They designed the preferences to generate a constant risk free rate. To a first approximation, it is conceivable that such a specification may yield a constant IMRS, thereby avoiding the sensitivity due to variations in the mean IMRS altogether. Campbell and Cochrane specify preferences as:

\[ U(c_t, X_t) = E_0 \sum_{t=0}^{\infty} \beta^t \left( \frac{(c_t - X_t)^{1-\sigma}}{1-\sigma} - 1 \right). \]

The IMRS is given by:

\[ \beta \left( \frac{S_{t+1}C_{t+1}}{S_tC_t} \right)^{-\sigma}, \]

where \( S_t \) is the surplus consumption ratio is given by \( S_t = \frac{C_t - X_t}{C_t} \) and \( C \) denotes aggregate consumption.

In equilibrium, \( c = C \). The evolution of the Habit stock is defined in terms of \( \ln(S_t) \):

\[ \ln(S_{t+1}) = (1-\phi)\ln(\bar{S}) + \phi\ln(S_t) + \lambda[\ln(S_t)](\ln(C_{t+1}) - \ln(C_t) - g) \]

where

\[ \lambda[\ln(S_t)] = \begin{cases} 
\frac{1}{\bar{S}} \sqrt{1 - 2(\ln(S_t) - \ln(\bar{S}))} - 1 & \ln(S_t) \leq S_{max} \\
0 & \ln(S_t) > S_{max}
\end{cases} \]

\[ \bar{S} = \kappa \sqrt{\frac{\sigma}{1-\phi}}. \]

The parameter \( \kappa \) is the standard deviation of consumption growth, while \( g \) is its mean. Note that the evolution of the habit stock depends on the parameters of the consumption growth process and hence will change in each subsample. With \( \beta = 0.89 \), \( \phi = 0.87 \) and \( \sigma = 1.08 \), the volatility in the IMRS exceeds the HJ bound as illustrated in panel a of Figure B1.

Despite the additional free parameters and the complex evolution of the habit stock, the Campbell-Cochrane model does not fair much better than either of the two models in Section IV.\(^8\)

Figure B2 repeats the bootstrap experiments from Section IV for the Campbell-Cochrane model. Panel a of the figure shows that the distance to the bound is generally negative, indicating that the model violated the HJ bound. In fact, the model violated the bound in 95.4% of the samples, an inconsequential improvement over the models in Section IV. Panel b of the figure shows that the mean IMRS of the Campbell-Cochrane model varies substantially relative to the mean IMRSs of the models in Section IV (see Figure 7).

\(^8\) We computed the equivalent of Table 3 for the Campbell-Cochrane model. The model misses the HJ bound in every subsample.
Figure B1: The Campbell-Cochrane Model

Figure B2: Bootstrap Experiment for the Campbell-Cochrane Model

a: Distance to the Bound  
b: IMRS Dispersion