Intermediate Macroeconomics  
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Handout 8  
Chapter 6: Long-run Economic Growth: The Solow Model

I. The Solow model provides a helpful starting point for understanding the nature of the  
dynamics of economic growth, and for considering a number of specific issues and  
questions related to growth. These include the following.

A. The dynamics of the simultaneous determination of capital accumulation and  
   economic growth.

B. The relationship between a nation’s long-run standard of living and fundamental  
   variables characterizing that nation’s economy, such as: saving rate, rate of growth in the  
   labor force, and rate of technical progress.

C. Will a nation’s rate of growth stabilize, accelerate, or stop?

D. Is the nature of economic growth such that ultimately, poorer countries catch up  
   with richer countries, in terms of standard of living?

II. The simplest version of the Solow model: maintained assumptions

A. The economy is closed.

B. There is no government.

C. The labor force is a fixed fraction of the population, and the population grows at a  
   constant rate $n$. This implies that the labor force also grows at the constant rate $n$. This in  
turn implies that the growth rate of consumption per worker equals the growth rate of  
consumption per capita.

D. The labor force is fully employed.

III. The structure of the model

A. The production function.

   1. Output $Y_t$ is produced by means of capital $K_t$ and labor $N_t$. The production  
      function, in aggregate form, is written: $Y_t = F(K_t, N_t)$. For simplicity, we begin with $A_t$  
      = 1. (We shall bring back this productivity variable later.) It is assumed that there are  
diminishing marginal productivity of capital and of labor, (review the first part of Chapter  
3).

   2. The production function is assumed to be characterized by constant returns to  
      scale. (recall the discussion of constant returns to scale in class). Therefore, it can be  
   written in per-worker form: $y_t = f(k_t)$, where $y_t = \frac{Y_t}{N_t}$ is output per worker, and  

   $k_t = \frac{K_t}{N_t}$ is capital per worker, though we usually refer to it as the capital-labor ratio. The  
   per-worker form of the production function inherits the property of diminishing marginal  
productivity of the capital-labor ratio from the analogous property, (diminishing marginal  
productivity of capital), of the aggregate production function. See Figure 3.1 on p.63 and  
Figure 6.3 on p.216.
Example: \( Y_t = F(K_t, N_t) = K_t^{0.3} N_t^{0.7} \), with the productivity coefficient \( A \) set equal to 1.

Recall that this is an example of a Cobb-Douglas production function, and that when the exponents in a Cobb-Douglas production function add to one, the production function is characterized by constant returns to scale. Divide both sides by \( N_t \) to obtain: \( y_t = k_t^{0.3} \).

B. Labor force growth: \( N_{t+1} = (1 + n)N_t \), where \( n > 0 \) is the (net) labor force growth rate, which, as noted above, is also the population growth rate.

C. The depreciation rate for capital is a constant \( d \), where \( 0 < d < 1 \). This means that if an aggregate quantity of capital \( K_t \) is used for one period, at the end of that period there remains a quantity \( (1 - d)K_t \) of productive capital.

D. The aggregate saving function is given by: \( S_t = sY_t \), where the coefficient \( s \) is the saving rate, and satisfies \( 0 < s < 1 \). This is a very simple specification of the saving function, but it is perhaps the key assumption that makes the Solow model so tractable, and thereby so useful.

E. The dynamics of capital accumulation:

Net investment, which is the actual change in the capital stock, is given by gross investment, which is the production of new capital, minus depreciation, which is the quantity of capital lost to wear and tear:

\[
K_{t+1} - K_t = I_t - dK_t.
\]

IV. The general law of motion governing the capital-labor ratio

To transform everything into per-worker terms, and thereby transform Equation (1) into the law of motion for the capital-labor ratio, we do the following: (i) divide both sides by \( N_t \); (ii) remember that in this closed economy with no government, aggregate (gross) investment is equal to aggregate saving, so that per-worker gross investment is equal to per-worker saving: \( \frac{I_t}{N_t} = \frac{S_t}{N_t} = sY_t = sf(k_t) \); (iii) \( (1/N_t) = [(1 + n)/N_{t+1}] \), so that \( (K_{t+1}/N_t) = [(1 + n)K_{t+1}/N_{t+1}] = (1 + n)k_{t+1} \); (iv) add \( k_t \) to both sides; and finally (v) divide both sides by \( (1 + n) \). The result is:

\[
k_{t+1} = [(1 - d)k_t + sf(k_t)]/(1 + n).
\]

Equation (2) allows us, in principle, to answer the following sort of question, which is clearly a question that is of importance to policymakers who want to assess the growth prospects of an economy. Suppose that we have derived estimates for the parameters \( d, s, n \), and also have estimated the production function \( f(k) \). Starting from an initial value of the capital-labor ratio, \( k_i \), what will be the value of the capital-labor ratio, (from
which we can then use the model to derive corresponding values for output per worker and consumption per worker), a decade from now? Two decades from now? 50 years from now?

Of course, for time periods measured in decades, the results gained from the sort of calculations suggested above will be somewhat speculative, and be subject to qualifications and revisions. However, recall: the Solow model is intended as a “starting point” for understanding the general nature of growth, for understanding the relation between capital accumulation and growth in output, and so on. Studying time paths generated by means of Equation (2) will surely strengthen such understanding.

Note: In order to carry out an exercise of this sort, or indeed, to make use of this model for any empirical applications, one must make use of an appropriate measure of capital. In particular, the sort of dollar measurement that appears on p. 61 for the U.S. is not the type that is needed. One must construct an appropriate (chain-weighted) index of the capital stock before putting the Solow model to work empirically.

Example: Suppose that one estimates the per-worker production function to be Cobb-Douglas with the productivity coefficient set equal to 1, and the exponent set equal to 0.3: $f(k_t) = k_t^{0.3}$. Suppose also, the other parameters of the model are estimated to be: $d = 0.1, n = 0.02, s = 0.2$. Then Equation (2) becomes: $k_{t+1} = [0.9k_t + 0.2k_t^{0.3}] / (1.02)$. We can then begin with an initial value for the capital-labor ratio for year 1, say, $k_1 = 1$, plug it into the righthand side of the equation in the preceding sentence, and calculate the capital-labor ratio for year 2, $k_2 = 1.078$. This value is then plugged into the righthand side to generate the value for year 3, $k_3 = 1.133$, and so on.

V. Steady States

A. The Solow model predicts that, in the absence of (ongoing) technological progress, the economy reaches a steady state in the long run. In a steady state, all aggregate variables – labor, capital, output, saving, investment, and consumption – grow at the same rate, namely, the rate at which the labor force grows.

B. The relationship defining the steady state can be derived by going a little further with the sequence of steps on p. 2 above. However, the goal on p. 2 is to derive the relationship that characterizes the full dynamic path generated by the Solow model. When thinking about steady states, we want to restrict attention to the long-run “destination” of that path.

1. In steady state, the aggregate capital stock grows just fast enough so that (depreciation having been covered) each new worker is equipped just as well as existing workers.

2. This will occur just when, each period, the actual rate of growth in the aggregate capital stock, (i.e. the net rate of growth, depreciation having been covered), is equal to the actual rate of growth (net rate of growth) in the aggregate number of workers:
\[
\frac{K_{t+1} - K_t}{K_t} = \frac{I_t - dK_t}{K_t} = n,
\]
so that:

\[
I_t = (n + d)K_t.
\]

As above, making use of the fact that this economy is closed, and the fact that it has no government, aggregate gross investment is equal to aggregate saving. Recall that we have taken the saving function to have the simple form: \( S_t = sY_t \).

Finally, although we could carry on the analysis with everything in aggregate terms, it is much easier, and also instructive – given that all of the aggregate variables grow at the same rate as the labor force – to transform everything into per-worker terms. Hence, divide through in Equation (4) by the aggregate number of workers \( N_t \); divide through in the saving function by the aggregate number of workers; and substitute saving per worker for gross investment per worker on the lefthand side, to gain the relationship that defines the steady state capital-labor ratio:

\[
sf(k) = (n + d)k.
\]

Once we obtain the steady-state capital-labor ratio, \( k = k^* \), we can then derive the values of all of the other per-worker variables:

\[
\left(\frac{Y_t}{N_t}\right)^* = y^* = f(k^*)
\]
\[
\left(\frac{S_t}{N_t}\right)^* = \left(\frac{I_t}{N_t}\right)^* = sy^* = sf(k^*)
\]
\[
\left(\frac{C_t}{N_t}\right)^* = (1 - s)y^* = (1 - s)f(k^*)
\]

Example: Suppose that \( d = 0.1, n = 0.02, s = 0.2, f(k) = k^{0.3} \). Substituting into (5), we obtain: \( 0.2k^{0.3} = 0.12k \), which implies that \( k^* = 2.076 \). Then: \( y^* = (2.076)^{0.3} = 1.245 \), and one can easily go on to compute the corresponding values of steady-state consumption per worker, saving per worker, and investment per worker.

VI. It is important that you understand the content of Figures 6.3 and 6.5-6.9, along with that of Summary 8 in the text. (We shall return to Figure 6.4.)

A. If the economy starts at \( k_1 < k^* \), or at \( k_2 > k^* \), it will follow a growth path such that it eventually stabilizes at \( k^* \). Explain why.
B. A higher saving rate implies a higher steady-state capital-labor ratio; and higher corresponding steady-state output per worker, investment per worker, and saving per worker. Explain why. (Note that consumption per worker is missing from this list. We shall come back to this.)

C. A higher population growth rate implies a lower steady-state capital-labor ratio; and lower corresponding steady-state output per worker, investment per worker, saving per worker, and consumption per worker. Explain why.

Upon first thought, this result might suggest that a policy be implemented that controls population growth. However, as the authors discuss on p.223-224 in the text, there are some important issues not taken into account in the Solow model. Perhaps the most important issues omitted stem from demographics.

D. A one-time productivity increase implies a higher steady-state capital-labor ratio; and higher corresponding steady-state output per worker, investment per worker, saving per worker, and consumption per worker. Explain why.

As discussed on p.225 in the text, a productivity increase raises the steady-state output per worker and consumption per worker in two ways. Be sure to understand this.

E. The Golden Rule capital-labor ratio

1. As indicated briefly above, steady-state consumption per worker is given by:

   \[ c^* = (1 - s) f(k^*) = f(k^*) - (n + d)k^*. \]

2. Notice that an increase in the steady-state capital-labor ratio has two opposing effects on consumption per worker in steady state. On the one hand, when the steady-state capital-labor ratio increases, output per worker in steady state increases, implying that there is more for both saving per worker and consumption per worker. On the other hand, when the steady-state capital-labor ratio increases, more investment per worker (= saving per worker) is required to cover depreciation and to equip new workers, and this leaves less left for consumption per worker.

   When the steady-state capital-labor ratio is quite low, the first effect dominates. That is, the steady-state capital-labor ratio, and corresponding steady-state output per worker, are sufficiently low, so that an increase in the capital-labor ratio would be sustainable in terms of additional investment per worker (= saving per worker), while at the same time allowing more consumption per worker.

   When the steady-state capital-labor ratio is already quite high, the second effect dominates: the amount of steady-state investment per worker required is already too high. What is the meaning of “already too high?” The meaning is this: (i) the amount of steady-state investment per worker (= saving per worker) needed to sustain the current capital-labor ratio (i.e. capital per worker) is already so high, that (ii) yes, indeed, a further increase in the steady-state capital-labor ratio would increase steady-state output per worker; but (iii) this higher capital-labor ratio would require so much additional investment per worker (= saving per worker) to support it, that (iv) consumption per worker would actually fall. From the perspective of increasing consumption per worker,
the economy at such a starting point would be better off with a lower steady-state capital-labor ratio; one where the saving per worker needed to sustain that steady-state capital-labor ratio leaves more output per worker available to become consumption per worker. It is true that this would lower steady-state output per worker, but the needed saving per worker would also be lowered; lowered to such an extent that consumption per worker would be increased.

3. Golden Rule capital-labor ratio: the capital-labor ratio $k_G$ that maximizes consumption per worker in steady state.

   (a) The fundamental reason for the fact that a country can have too high a steady-state capital-labor ratio, (in the absence of technological progress – see below), is the fact of diminishing marginal productivity of capital. Each additional unit of output per worker (in steady state) requires greater and greater amounts of capital per worker (in steady state), to such an extent that further additions to capital per worker are not sustainable without reducing consumption per worker (in steady state). There would be simply too much additional investment per worker = saving per worker required (in steady state).

   (b) As the authors suggest, (p.219), some careful research has been done supporting the claim that no country has yet reached its Golden Rule steady state. They also state that henceforth, in their analysis, they assume that the capital-labor ratio is below the Golden Rule capital-labor ratio. Keep this in mind when reading the material in this handout and corresponding material in the text.

4. Much of the foregoing discussion of the Golden Rule capital-labor ratio can be summed up, (holding $(n + d)$ fixed; and the production function and corresponding $MPK$ function fixed; but with the values of these latter two changing when $k$ changes), by the equation:

$$MPK = (n + d).$$

Explain how this equation sums up the benefit-cost considerations related to the nature of the Golden Rule capital-labor ratio.

F. An intergenerational issue

Intergenerational issues arise when studying economic dynamics. We have already come across an example in the context of population growth. (Indeed, such issues can be raised in the context of Ricardian equivalence, e.g. see Handout 6 (cont’d.).) The present discussion of the Golden Rule capital-labor ratio provides a good context within which to consider another example.

VII. Technical progress and ongoing improvement in the standard of living

A. The Solow model implies that growth in per-worker output and per-worker consumption will stabilize, approaching zero in the long run, as the economy approaches a steady state.

B. As discussed in relation to VI.C.2.-4. above, a one-time increase in the saving rate; a one-time decrease in the population growth rate; or a one-time increase in productivity each increases the steady-state capital-labor ratio, the steady-state output per worker, and,
provided the saving rate is below the (unique value) that would yield the Golden Rule capital-labor ratio, the steady-state level of consumption per worker. Because of the relation we have assumed between the labor force and the total population, we can use the level of consumption per worker as a direct indicator of standard of living. Hence, we can conclude, at least as a starting point on the basis of the Solow model, that each of the one-time changes above implies a one-time improvement in the standard of living.

There is an upper limit to how much the population in a society is willing to save. Also, there is, one might argue, a lower limit to how long or how low a population growth rate might fall. Hence, for these two sorts of change, the Solow model implies that for any given economy, there is a maximum feasible (steady-state) standard of living. However, having examined the effect of a one-time increase in productivity above, we can see that, other things equal, the Solow model also implies that if, but only if, there are ongoing increases in productivity, that is, ongoing technological progress, then there will be ongoing improvement in a society’s standard of living in the long run. Accumulation of physical and human capital, (for a given state of “embodied” technology), are also necessary, but are not enough without ongoing technological progress.

While the Solow model is a rudimentary model, this implication seems to be common to many, more complex models as well. The engine of growth in the long run is increasing productivity, where this should be construed in a broad sense, to include ongoing improvements in productivity of human capital, physical capital, “organizational” capital, and so on.

VIII. Convergence

A. The question is raised at the beginning of this handout as to whether, by means of the engine of economic growth, (currently) poorer countries will catch up to (currently) richer ones. The Solow model does predict convergence in standard of living for any pair of countries if the only difference between the two is that one has a lower initial capital-labor ratio than the other. That is, the Solow model predicts what is known as unconditional convergence.

Here, think of both countries as not yet being in steady state, similar to the situation in the example at the end of IV above, with the poor country’s initial capital-labor ratio lower than the rich country’s, and with the initial capital-labor ratio for each country being below the steady state: \( k_{1}^{\text{poor}} < k_{1}^{\text{rich}} < k^{*} \), where the particular \( k^{*} \) that is relevant is determined by the saving rate, the population growth rate, and the depreciation rate, and also by the type of production function, which in the example at the end of IV above is Cobb-Douglas, with \( A = 1 \). (There must be no ongoing productivity growth if the two countries are to approach a particular steady state.) If you trace out the dynamic paths of the two countries, the poor country’s dynamic path will exhibit increments in the capital-labor ratio that are larger than the increments for the rich country. Gradually, the distance between the capital-labor ratio of the poor country and that of the rich country will shrink, as they both approach the same steady state \( k^{*} \).

B. However, if two countries differ in other respects, (i.e. the values of \( n, d, s, A \), or the production function \( f \)), in addition to initial capital-labor ratio, then the Solow model predicts that the rich country and the poor country will not converge to the same steady state. However, the simplest version of the Solow model applies to a closed economy. When one considers an open economy, other possibilities open up for both (i) factor input
accumulation, (through international capital markets); and (ii) technology transfer, which can contribute to a faster rate of improvement in productivity in some of the poorer countries. Here we leave this issue of convergence.

IX. Endogenous growth theory

The authors’ treatment of endogenous growth theory gives too little emphasis to the most important aspect of the contribution of endogenous growth theory to the theory of economic growth. This aspect is the explicit incorporation into growth models of the economic incentives that lie behind research and development, innovation and improvements in productivity, the accumulation of human and physical capital, and so on. Those incentives include such things as the pursuit of higher wages, the pursuit of increases in returns to capital investment, etc. Such incentives lead to the accumulation of quantities of inputs, and, as indicated above, even more importantly for growth, ongoing improvements in productivity.

X. Government policies and long-run living standards

Of course, for both poor countries and rich countries, but especially pressing for poor countries, there is a role for government to play in promoting economic growth. As the authors point out in the last section of the chapter, of particular importance are: (i) the provision of adequate infrastructure; (ii) a role for government in the provision of health programs; and then along with these first two, (iii) a role for government in the development of human capital, through adequate opportunities for education, and through appropriate policies that encourage, rather than discourage, entrepreneurial activity, including entrepreneurial risk-taking; and (iv) a role for government in helping to provide a framework that encourages research and development activity. This last is a source of ongoing controversy, in particular with respect to patents policy.